# MATH 122

Ann Clifton [1]

[1] University of South Carolina, Columbia, SC USA

Calculus for Business Administration and Social Sciences

# Outline



5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

1. 5.1: Distance and Accumulated Change
   - Constant Functions
   - Linear Functions
   - Non-Linear Functions
   - Right Endpoint Estimates
   - Left Endpoint Estimates
   - Partitions
   - Left- and Right-Hand Sums
   - Applying Our Method

# Constant Functions



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Suppose a car is traveling at 60 miles per hour for 2 hours.

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Constant Functions

Suppose a car is traveling at 60 miles per hour for 2 hours. How far did the car go?

# Constant Functions



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Suppose a car is traveling at 60 miles per hour for 2 hours. How far did the car go?

This is easy:

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Suppose a car is traveling at 60 miles per hour for 2 hours. How far did the car go?

This is easy:

$$60 \, \frac{\text{miles}}{\text{hour}} \cdot 2 \text{ hours} =$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Constant Functions

Suppose a car is traveling at 60 miles per hour for 2 hours. How far did the car go?

This is easy:

$$60\ \frac{\text{miles}}{\text{hour}} \cdot 2\ \text{hours} = 120\ \text{miles}.$$

# Constant Functions



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

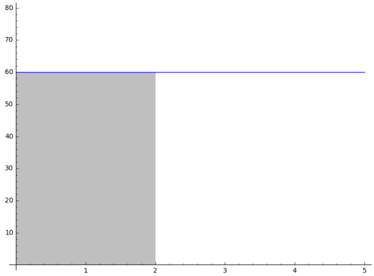
Geometrically, this is the area under the constant curve $y(t) = 60$ between $t = 0$ and $t = 2$:

# Constant Functions



Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Geometrically, this is the area under the constant curve $y(t) = 60$ between $t = 0$ and $t = 2$:



Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

This says that under constant velocity, $v$, the position of the car, $s(t)$, relative to the starting point at time $0 \leq t$ is just

$$s(t) = v \cdot t.$$

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

According to Car and Driver, a 2006 Bugatti Veyron is capable of an acceleration of $11.59\,\mathrm{m}/\mathrm{s}^2$. Assume the car starts at rest and accelerates at this constant rate.

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

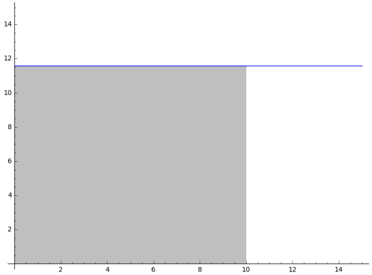
According to Car and Driver, a 2006 Bugatti Veyron is capable of an acceleration of $11.59\,\mathrm{m}/\mathrm{s}^2$. Assume the car starts at rest and accelerates at this constant rate.

By the observation in the last example, we can compute the velocity at time $t$ as the area under the constant curve $y(t) = 11.59$:

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

According to Car and Driver, a 2006 Bugatti Veyron is capable of an acceleration of $11.59\,\mathrm{m}/\mathrm{s}^2$. Assume the car starts at rest and accelerates at this constant rate.

By the observation in the last example, we can compute the velocity at time $t$ as the area under the constant curve $y(t) = 11.59$:



# Linear Functions



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

The velocity is linear: $v(t) = 11.59 \cdot t$. Hence the position, $s(t)$, is the area under the velocity curve:

The velocity is linear: $v(t) = 11.59 \cdot t$. Hence the position, $s(t)$, is the area under the velocity curve:

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Therefore the position at time $t$ is:

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Therefore the position at time $t$ is:

$$s(t) \quad =$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Therefore the position at time $t$ is:

$$s(t) = \frac{1}{2}v(t) \cdot t$$

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Therefore the position at time $t$ is:

$$\begin{aligned} s(t) &= \frac{1}{2}v(t) \cdot t \\ &= \frac{1}{2}(11.59 \cdot t) \cdot t \end{aligned}$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Therefore the position at time $t$ is:

$$\begin{aligned} s(t) &= \frac{1}{2}v(t)\cdot t \\ &= \frac{1}{2}(11.59\cdot t)\cdot t \\ &= \frac{11.59}{2}t^2. \end{aligned}$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

What happens when the area is not a nice geometric object?

# Non-Linear Functions



What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

# Non-Linear Functions

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

MATH 122

CLIFTON

5.1: DISTANCE AND ACCUMULATED CHANGE

CONSTANT FUNCTIONS

LINEAR FUNCTIONS

NON-LINEAR FUNCTIONS

RIGHT ENDPOINT ESTIMATES

LEFT ENDPOINT ESTIMATES

PARTITIONS

LEFT- AND RIGHT-HAND SUMS

APPLYING OUR METHOD

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

$$\frac{30-20}{2-0} =$$

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

$$\frac{30-20}{2-0}=5$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

$$\frac{30-20}{2-0} = 5 \text{ and}$$

# Non-Linear Functions



What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

$$\frac{30-20}{2-0} = 5 \text{ and } \frac{50-48}{10-8}$$

Math 122

Clifton

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

This is clearly not linear:

$$\frac{30-20}{2-0} = 5 \text{ and } \frac{50-48}{10-8} = 1.$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Non-Linear Functions

What happens when the area is not a nice geometric object?

We can fit a curve to these points:

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

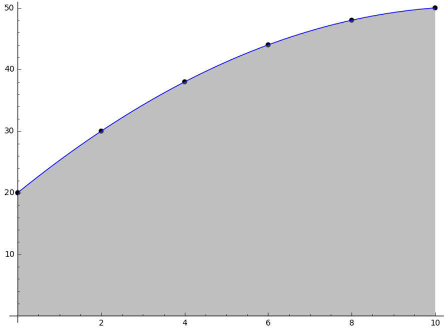
Partitions

Left- and Right-Hand Sums

Applying Our Method

What happens when the area is not a nice geometric object?

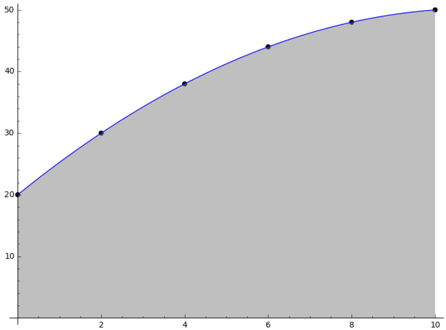
We can fit a curve to these points:



# Non-Linear Functions

What happens when the area is not a nice geometric object?

We can fit a curve to these points:




How do we compute the area of the shaded region?

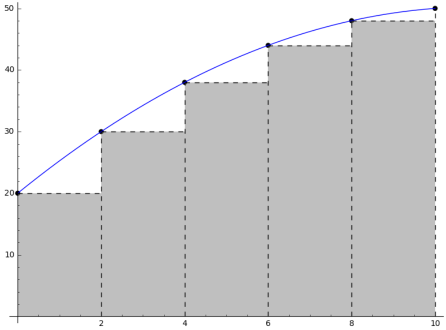
Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

We could assume constant velocity between the two points and estimate.

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

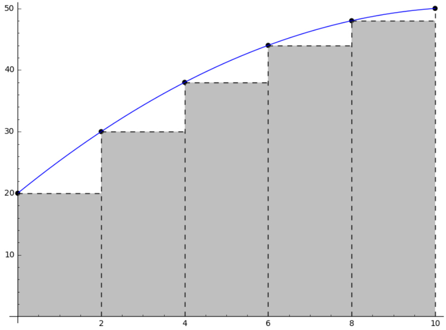
We could assume constant velocity between the two points and estimate. Say we assume the velocity is the velocity at the left endpoint:

We could assume constant velocity between the two points and estimate. Say we assume the velocity is the velocity at the left endpoint:



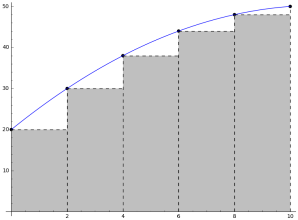
# Non-Linear Functions

We could assume constant velocity between the two points and estimate. Say we assume the velocity is the velocity at the left endpoint:



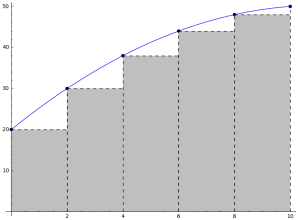

This is an underestimate of the area.

# Non-Linear Functions



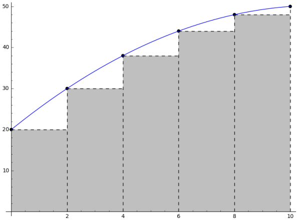

- Each rectangle has width 2.

# Non-Linear Functions



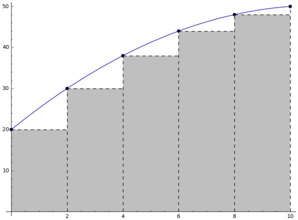

- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.

# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.
- Our area estimate is:

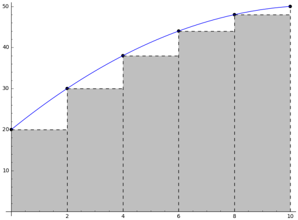
# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.
- Our area estimate is:

$$2(20 + 30 + 38 + 44 + 48) \quad =$$

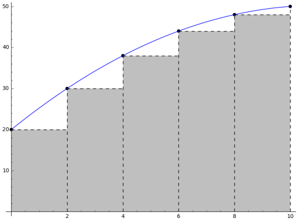
# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.
- Our area estimate is:

$$2(20+30+38+44+48) = 2(180)$$

# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.
- Our area estimate is:

$$\begin{aligned} 2(20+30+38+44+48) &= 2(180) \\ &= 360 \text{ feet.} \end{aligned}$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

We could also assume the velocity is the velocity at the right endpoint:

MATH 122

CLIFTON

5.1: DISTANCE AND ACCUMULATED CHANGE

CONSTANT FUNCTIONS

LINEAR FUNCTIONS

NON-LINEAR FUNCTIONS

RIGHT ENDPOINT ESTIMATES

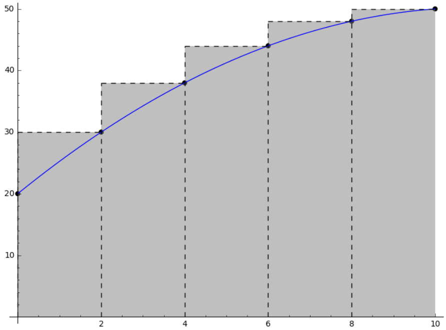
LEFT ENDPOINT ESTIMATES

PARTITIONS

LEFT- AND RIGHT-HAND SUMS

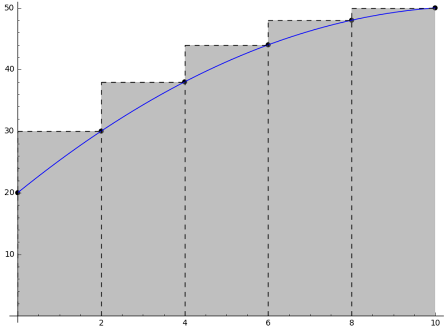
APPLYING OUR METHOD

We could also assume the velocity is the velocity at the right endpoint:



MATH 122
CLIFTON
5.1: DISTANCE AND ACCUMULATED CHANGE
CONSTANT FUNCTIONS
LINEAR FUNCTIONS
NON-LINEAR FUNCTIONS
RIGHT ENDPOINT ESTIMATES
LEFT ENDPOINT ESTIMATES
PARTITIONS
LEFT- AND RIGHT-HAND SUMS
APPLYING OUR METHOD

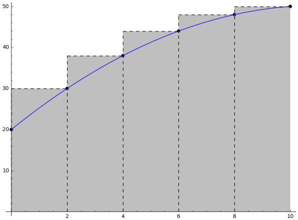
We could also assume the velocity is the velocity at the right endpoint:




This is an overestimate of the area.

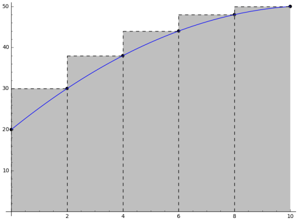
# Non-Linear Functions





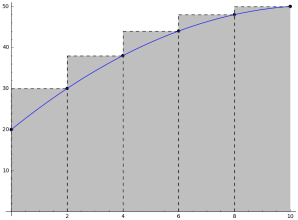

- Each rectangle has width 2.

# Non-Linear Functions



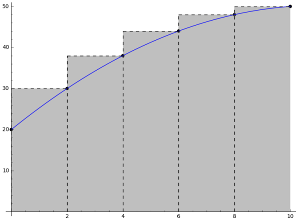

- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.

# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.
- Our area estimate is:

# Non-Linear Functions

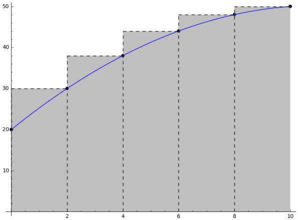



- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.
- Our area estimate is:

$$2(30 + 38 + 44 + 48 + 50) \quad =$$

# Non-Linear Functions

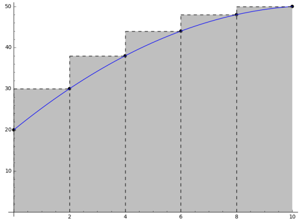





- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.
- Our area estimate is:

$$2(30+38+44+48+50) \quad = \quad 2(210)$$

# Non-Linear Functions




- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.
- Our area estimate is:

$$
\begin{aligned}
2(30 + 38 + 44 + 48 + 50) &= 2(210) \\
&= 420 \text{ feet.}
\end{aligned}
$$

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

This tells us:

- The distance traveled is **at least** 360 feet.

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.
- The distance traveled must be somewhere between these two.

This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.
- The distance traveled must be somewhere between these two.
- The average of these estimates is

$$\frac{420 + 360}{2} = 390$$

feet, which gives a better estimate.

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.
- The distance traveled must be somewhere between these two.
- The average of these estimates is

  $$\frac{420 + 360}{2} = 390$$

  feet, which gives a better estimate.

Can we do better?

# Non-Linear Functions



This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.
- The distance traveled must be somewhere between these two.
- The average of these estimates is

$$\frac{420 + 360}{2} = 390$$

feet, which gives a better estimate.

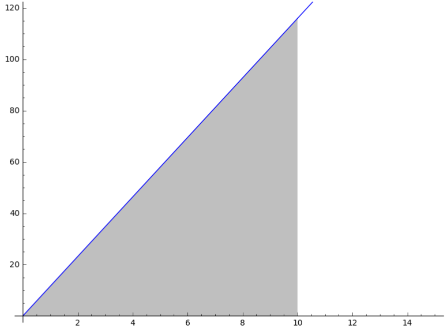
Can we do better? If so, how?

# RIGHT ENDPOINT ESTIMATES

We'll use the old linear velocity example, $v(t) = 11.59t$, to analyse these methods:

# Right Endpoint Estimates

We'll use the old linear velocity example, $v(t) = 11.59t$, to analyse these methods:



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$.

# Right Endpoint Estimate



Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

# Right Endpoint Estimate



Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2}v(t) \cdot t.$$

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2}v(t) \cdot t.$$

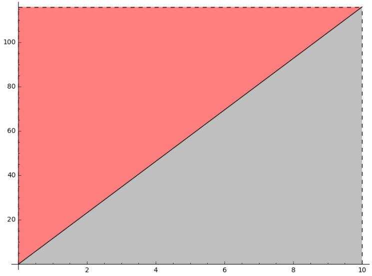
Our estimate is quite bad:

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2}v(t) \cdot t.$$

Our estimate is quite bad:

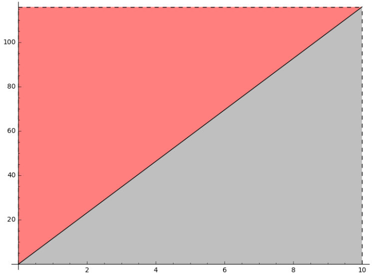


# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2} v(t) \cdot t.$$

Our estimate is quite bad:



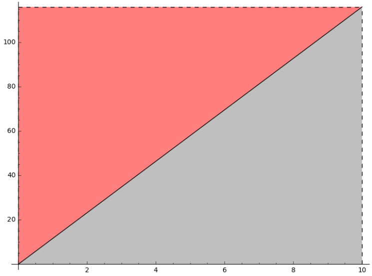

- Red is the error.

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2} v(t) \cdot t.$$

Our estimate is quite bad:



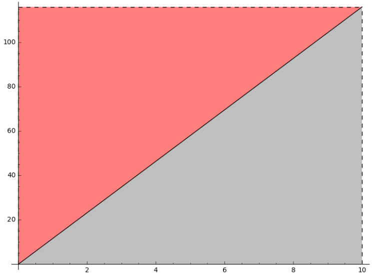

- Red is the error.
- Grey is the area.

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2}v(t) \cdot t.$$

Our estimate is quite bad:



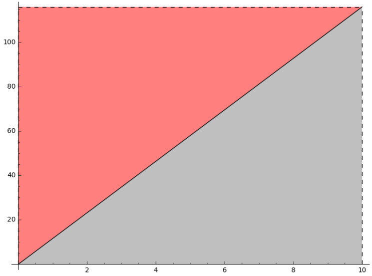

- Red is the error.
- Grey is the area.
- The estimate for the area is the sum of the red and grey areas.

# Right Endpoint Estimate

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2} v(t) \cdot t.$$

Our estimate is quite bad:




- Red is the error.
- Grey is the area.
- The estimate for the area is the sum of the red and grey areas.
- The error is equal to the actual area!

# Three Equidistant Points

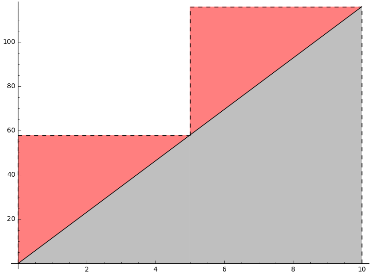


If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:

# Three Equidistant Points

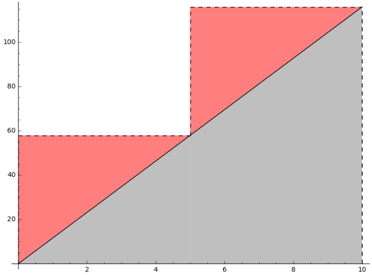


If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:



# Three Equidistant Points

If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:

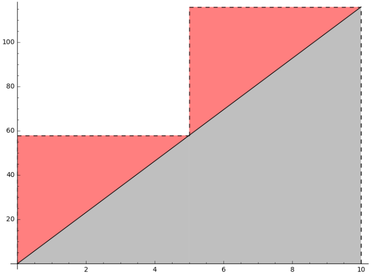



- Visibly, this is a better estimate.

# Three Equidistant Points



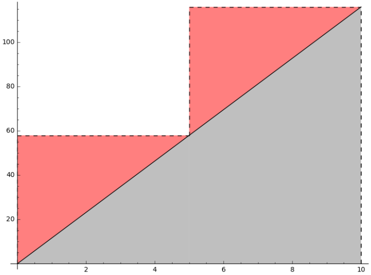
If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:




- Visibly, this is a better estimate.
- The error is the area of the two red triangles.

# Three Equidistant Points

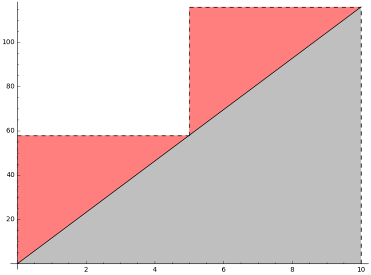
If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:




- Visibly, this is a better estimate.
- The error is the area of the two red triangles.
- Both have base length $\frac{t}{2}$; here $t = 10$.

# Three Equidistant Points

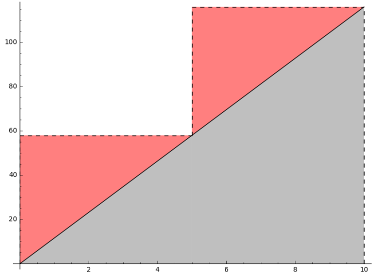
If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:




- Visibly, this is a better estimate.
- The error is the area of the two red triangles.
- Both have base length $\frac{t}{2}$; here $t = 10$.
- The height of the left triangle is $v\left(\frac{t}{2}\right)$.

# Three Equidistant Points

If we try three equidistant points, 0, $\frac{t}{2}$, and $t$, then we get:




- Visibly, this is a better estimate.
- The error is the area of the two red triangles.
- Both have base length $\frac{t}{2}$; here $t = 10$.
- The height of the left triangle is $v\left(\frac{t}{2}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{t}{2}\right)$.

# Three Equidistant Points (cont.)

So, the total error is:

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

So, the total error is:

$$\frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)\right]\frac{t}{2}+$$

# Three Equidistant Points (Cont.)



So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} + \frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2}$$

# Three Equidistant Points (cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} + \frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} \quad = \quad \frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right) + v\left(\frac{t}{2}\right)\right]\frac{t}{2}$$

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method


So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)\right]\frac{t}{2}+\frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} &= \frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)+v\left(\frac{t}{2}\right)\right]\frac{t}{2}\\ &= \frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

# Three Equidistant Points (cont.)

So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)\right]\frac{t}{2}+\frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} &= \frac{1}{2}\left[v(t)-v\left(\frac{t}{2}\right)+v\left(\frac{t}{2}\right)\right]\frac{t}{2}\\ &= \frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

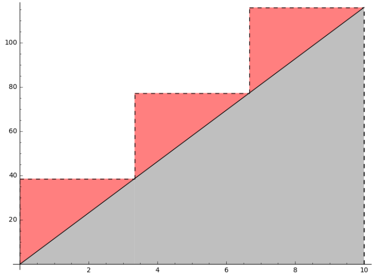
By adding one more point, we've reduced the error by a factor of two!

# Four Equidistant Points (Cont.)

If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:

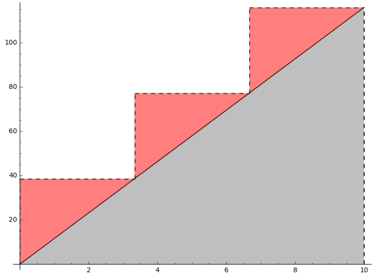
# Four Equidistant Points (cont.)

If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:



# Four Equidistant Points (cont.)

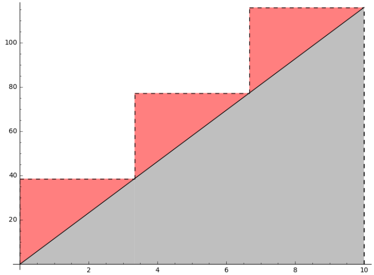
If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:




- Visibly, this is an even better estimate.

# Four Equidistant Points (Cont.)

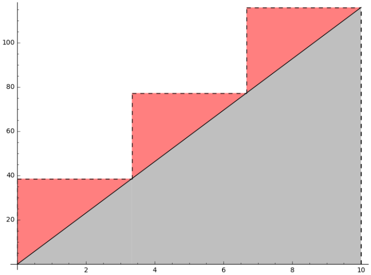
If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:




- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.

# Four Equidistant Points (Cont.)

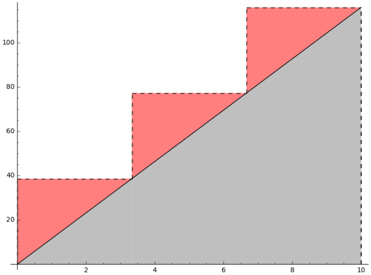
If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:




- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.

# Four Equidistant Points (cont.)

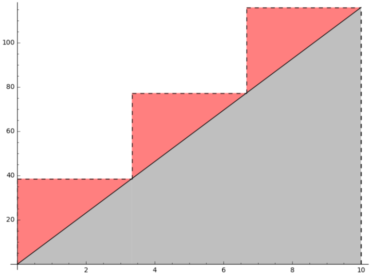
If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:




- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.
- The height of the middle triangle is $v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)$.

# Four Equidistant Points (Cont.)

If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and $t$, then we get:




- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.
- The height of the middle triangle is $v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{2t}{3}\right)$.

# Four Equidistant Points (Cont.)



So, the total error is:

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Four Equidistant Points (cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} +$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method


So, the total error is:

$$\frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}\left[v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)\right]\frac{t}{3}+$$

# Four Equidistant Points (Cont.)

So, the total error is:

$$\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3}$$

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

So, the total error is:

$$\frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}\left[v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} = \frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)+v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)+v\left(\frac{t}{3}\right)\right]\frac{t}{3}$$

# Four Equidistant Points (Cont.)



So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}\left[v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} &= \frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)+v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)+v\left(\frac{t}{3}\right)\right]\frac{t}{3}\\ &= \frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

# Four Equidistant Points (Cont.)

So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}\left[v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)\right]\frac{t}{3}+\frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} &= \frac{1}{2}\left[v(t)-v\left(\frac{2t}{3}\right)+v\left(\frac{2t}{3}\right)-v\left(\frac{t}{3}\right)+v\left(\frac{t}{3}\right)\right]\frac{t}{3}\\ &= \frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

By using four points, we've reduced the initial error by a factor of three!

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# $n + 1$ Equidistant Points

If we use $n + 1$ equidistant points,

# $n+1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,$$

# $n+1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},$$

# $n + 1$ Equidistant Points



If we use $n + 1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},$$

# $n + 1$ Equidistant Points



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

If we use $n + 1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,$$

# $n+1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},$$

# $n + 1$ Equidistant Points



If we use $n + 1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

# $n+1$ Equidistant Points



If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles.

# $n + 1$ Equidistant Points

If we use $n + 1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

# $n+1$ Equidistant Points



If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,

# $n+1$ Equidistant Points



If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,

# $n + 1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

# $n+1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

**Remark 1**

Note that $v(t_0) = v(0) = 0$.

# $n + 1$ Equidistant Points



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Adding up the areas of each of the triangles, we get the total error:

# $n+1$ Equidistant Points

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Adding up the areas of each of the triangles, we get the total error:

$\frac{1}{2}[$

# $n+1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$\frac{1}{2}\left[v(t) - v(t_{k-1}) +\right.$

# $n + 1$ Equidistant Points


Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method


Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \right.$$

# $n+1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + \right.$$

5.1: DISTANCE AND ACCUMULATED CHANGE
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + \right.$$

# $n + 1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n}$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} \quad = \quad \frac{1}{2}v(t)\cdot\frac{t}{n}$$

# $n + 1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned} \frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n} \\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right). \end{aligned}$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n} \\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

Therefore, if we use $n+1$ equidistant points, we have overestimated the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Left Estimate

The situation for a left endpoint estimate is symmetric:

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

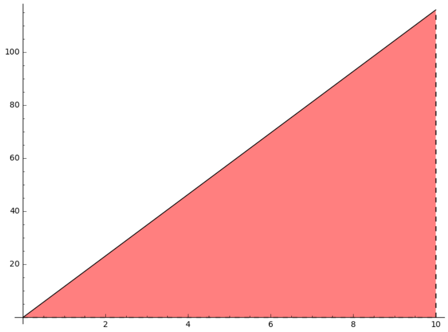
Partitions

Left- and Right-Hand Sums

Applying Our Method


The situation for a left endpoint estimate is symmetric:

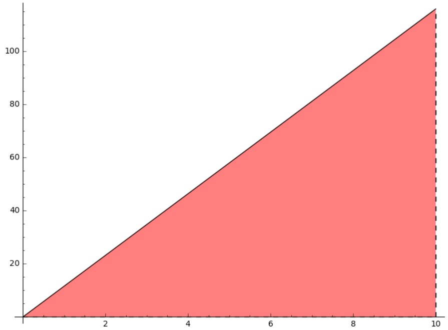
2 Equidistant Points:



# Left Estimate

The situation for a left endpoint estimate is symmetric:

2 Equidistant Points:

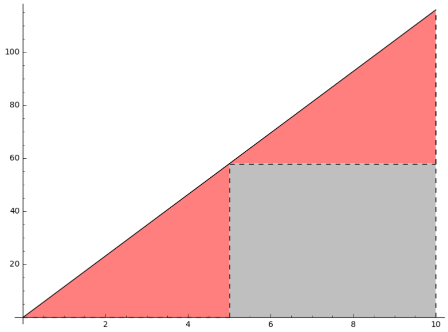



Our Estimate for the area here is **zero**. We have **underestimated** the area by $\frac{1}{2}v(t) \cdot t$.

# Left Estimate

The situation for a left endpoint estimate is symmetric:

3 Equidistant Points:



Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

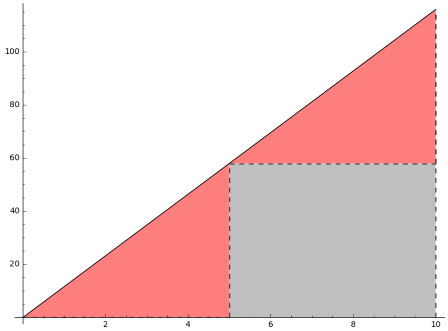
Partitions

Left- and Right-Hand Sums

Applying Our Method


The situation for a left endpoint estimate is symmetric:

3 Equidistant Points:

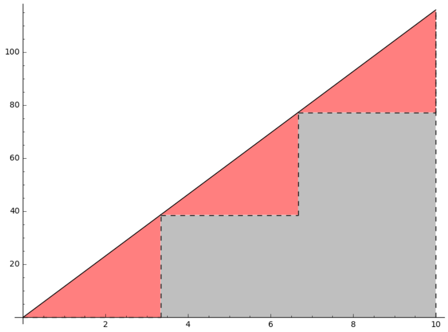



We have **underestimated** the area by $\frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right)$.

# Left Estimate

The situation for a left endpoint estimate is symmetric:

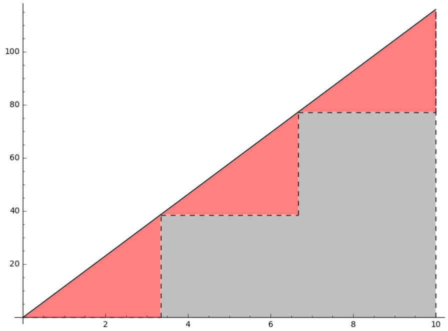
4 Equidistant Points:



# Left Estimate

The situation for a left endpoint estimate is symmetric:

4 Equidistant Points:




We have **underestimated** the area by $\frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right)$.

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

# Left Estimate

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,$$

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0, \; t_1 = \frac{t}{n},$$

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},$$

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,$$

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},$$

# Left Estimate


Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method


By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

# Left Estimate



By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles.

# Left Estimate



By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

# Left Estimate



By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,

# Left Estimate



By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

# Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of $n$ triangles. The $k^{\text{th}}$ triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

**Remark 2**

Note that $v(t_0) = v(0) = 0$.

# $n+1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

# $n + 1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$\frac{1}{2}[$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + \right.$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# $n + 1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \right.$$

# $n + 1$ Equidistant Points



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots +\right.$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + \right.$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n}$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} = \frac{1}{2}v(t)\cdot\frac{t}{n}$$

# $n+1$ Equidistant Points



Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned} \frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n} \\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right). \end{aligned}$$

# $n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned}\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n}\\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

Therefore, if we use $n+1$ equidistant points, we have **underestimated** the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# More Is Better

- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t=10$ is given by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right)=\frac{1}{n}\left(\frac{11.59}{2}100\right).$$

# More Is Better



- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t = 10$ is given by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right) = \frac{1}{n}\left(\frac{11.59}{2}100\right).$$

- This tells us that as $n$ becomes large, the error decreases. That is, the more points, the better the estimate!

# More Is Better



- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t = 10$ is given by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right) = \frac{1}{n}\left(\frac{11.59}{2}100\right).$$

- This tells us that as $n$ becomes large, the error decreases. That is, the more points, the better the estimate!
- As $n$ grows larger, the right estimate **decreases** towards the actual area and the left estimate **increases** towards the actual area.

# Right Error

# Left Error




5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Partitions of an Interval

To generalize our methods to non-linear curves, we introduce some notation.

## Definition 1

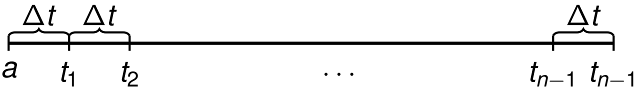
For a continuous function, $f$, on an interval $[a, b]$, a set of $n + 1$ equidistant points,

$$t_0 = a < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

is called a *partition* of $[a, b]$.

# Partitions and Estimates

These $n+1$ points are called a partition because they partition $[a, b]$ into $n$ smaller intervals of length $\Delta t$

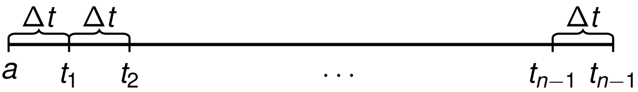



where

$$\Delta t = \frac{b-a}{n}.$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

These $n+1$ points are called a partition because they partition $[a, b]$ into $n$ smaller intervals of length $\Delta t$




where

$$\Delta t = \frac{b-a}{n}.$$

These $n$ smaller intervals form the bases of the rectangles we use to estimate the area under a curve.

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

**Definition 2**

Let $f$ be a continuous function on the interval $[a, b]$.

# Sums



## Definition 2

Let $f$ be a continuous function on the interval $[a, b]$. Given a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

# Sums

## Definition 2

Let $f$ be a continuous function on the interval $[a, b]$. Given a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

- The *Left-Hand Sum* is

$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-2})\Delta t + f(t_{n-1})\Delta t.$$

# Sums

## Definition 2

Let $f$ be a continuous function on the interval $[a, b]$. Given a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

- The *Left-Hand Sum* is

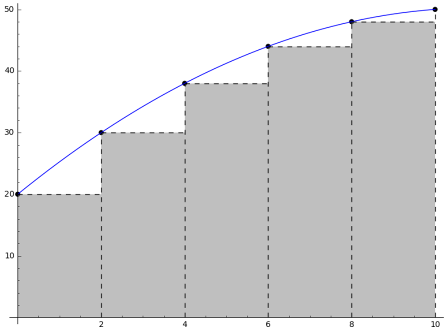
$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-2})\Delta t + f(t_{n-1})\Delta t.$$

- The *Right-Hand Sum* is

$$f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{n-1})\Delta t + f(t_n)\Delta t.$$

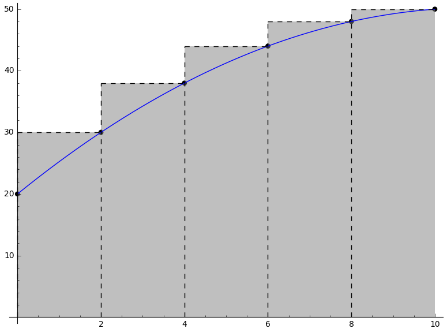
# Sums (Cont.)

The Left-Hand Sum underestimates the area under the curve:



# Sums (cont.)

The Right-Hand Sum overestimates the area under the curve:



# Sigma Notation

For ease of notation, we write the left-hand sum as

# Sigma Notation



For ease of notation, we write the left-hand sum as

$$\sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + \ldots + f(t_{n-1})\Delta t$$

# Sigma Notation



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

For ease of notation, we write the left-hand sum as

$$\sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + \ldots + f(t_{n-1})\Delta t$$

and we write the right-hand sum as

$$\sum_{i=1}^{n} f(t_i)\Delta t = f(t_1)\Delta t + \ldots + f(t_n)\Delta t.$$

Math 122
Clifton
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Sigma Notation

For ease of notation, we write the left-hand sum as

$$\sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + \ldots + f(t_{n-1})\Delta t$$

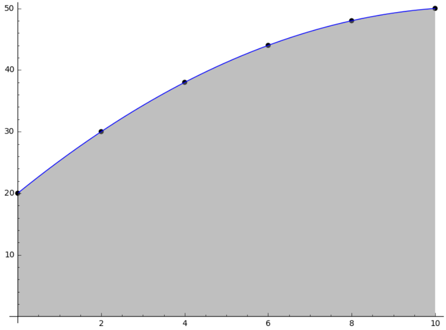
and we write the right-hand sum as

$$\sum_{i=1}^{n} f(t_i)\Delta t = f(t_1)\Delta t + \ldots + f(t_n)\Delta t.$$

The letter $i$ is the *index* of the summation and the letter $n$ is the *upper bound* of the summation. The $i = 0$ underneath the sigma, $\Sigma$, indicates the sum starts at 0 and the upper bound indicates when to stop.

# Generalizing Our Analysis

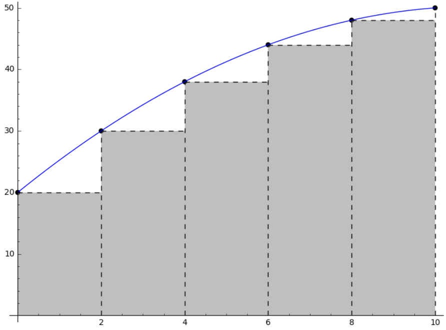
The entire point of our analysis of the linear velocity example was to improve our estimates for the non-linear curve



# Generalizing our Analysis

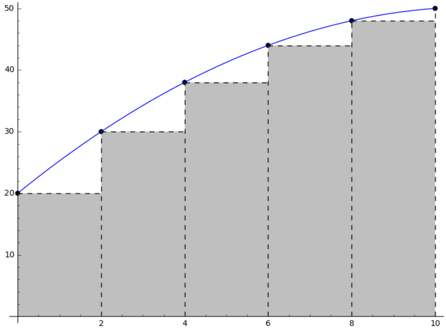


When we use a Left-Hand Sum, we can't necessarily write down the error explicitly because the error isn't quite a triangle:



# Generalizing our Analysis

When we use a Left-Hand Sum, we can't necessarily write down the error explicitly because the error isn't quite a triangle:




However, we can use differential calculus to get around this.

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Let $f$ be a continuous function. Recall that if we take $\Delta t$ sufficiently small, then we can use the Tangent Line Approximation,

$$f(t) \approx f'(a)(t - a) + f(a),$$

to ensure that $f$ is basically a line whenever $a \leq t \leq a + \Delta t$.

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Linearization for Left-Hand Sums (Cont.)

Say we want to find the area beneath a continuous curve, $f$, on the interval $[a, b]$.

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Linearization for Left-Hand Sums (Cont.)

Say we want to find the area beneath a continuous curve, $f$, on the interval $[a, b]$.

- We can control the size of $\Delta t$ by increasing the number of points in a partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

since

$$\Delta t = \frac{b-a}{n}.$$

# Linearization for Left-Hand Sums (Cont.)

Say we want to find the area beneath a continuous curve, $f$, on the interval $[a, b]$.

- We can control the size of $\Delta t$ by increasing the number of points in a partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

since

$$\Delta t = \frac{b - a}{n}.$$

- This means that if we use enough points,

$$f(t) \approx f'(t_i)(t - t_i) + f(t_i),$$

whenever $t_i \leq t \leq t_{i+1}$, and in particular

$$f(t_{i+1}) \approx f'(t_i)\Delta t + f(t_i).$$

# Linearization for Left-Hand Sums (Cont.)



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

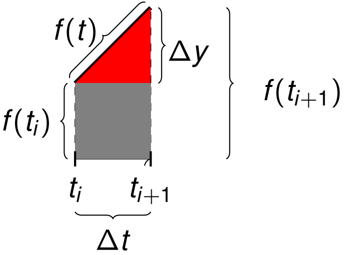
Left- and Right-Hand Sums

Applying Our Method

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:

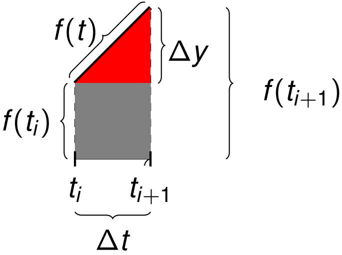
# Linearization for Left-Hand Sums (cont.)

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:



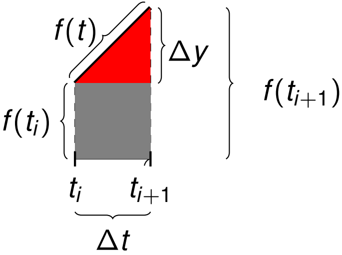
# Linearization for Left-Hand Sums (cont.)

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:




By our previous analysis, the Left-Hand Sum underestimates the area under $f$ on the interval $[t_i, t_{i+1}]$ by approximately

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:

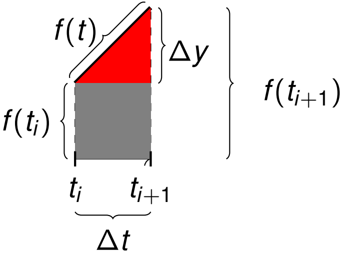



By our previous analysis, the Left-Hand Sum underestimates the area under $f$ on the interval $[t_i, t_{i+1}]$ by approximately

$$\frac{1}{2}\Delta y \Delta t$$

# Linearization for Left-Hand Sums (Cont.)

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:




By our previous analysis, the Left-Hand Sum underestimates the area under $f$ on the interval $[t_i, t_{i+1}]$ by approximately

$$\frac{1}{2}\Delta y \Delta t = \frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t.$$

# Linearization for Left-Hand Sums (cont.)

- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.

# Linearization for Left-Hand Sums (cont.)



- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

# Linearization for Left-Hand Sums (Cont.)

- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t$$

# Linearization for Left-Hand Sums (cont.)

- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

# Linearization for Left-Hand Sums (Cont.)



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Since $M - m$ is a fixed constant, this value goes to zero as $n$ becomes large!

# Linearization for Left-Hand Sums (cont.)



- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Since $M - m$ is a fixed constant, this value goes to zero as $n$ becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.

# Linearization for Left-Hand Sums (cont.)

- By our work in Chapter 4, $f$ attains a global maximum, $M$, and a global minimum, $m$, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Since $M - m$ is a fixed constant, this value goes to zero as $n$ becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.
- As we increase the number of points in our partition, the Left-Hand Sum **increases** towards the area under the curve.

# Left Sum



5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Linearization for Right-Hand Sums



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.

# Linearization for Right-Hand Sums

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

# Linearization for Right-Hand Sums

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t$$

# Linearization for Right-Hand Sums



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Linearization for Right-Hand Sums

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Again, as $M - m$ is a constant, this value goes to zero as $n$ becomes large!

# Linearization for Right-Hand Sums



- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Again, as $M - m$ is a constant, this value goes to zero as $n$ becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.

# Linearization for Right-Hand Sums

- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Again, as $M - m$ is a constant, this value goes to zero as $n$ becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.
- As we increase the number of points in our partition, the Right-Hand Sum **decreases** towards the area under the curve.

# Right Sum

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

# Our Distance Traveled Example

Recall that we started this excursion with the following question:

# Our Distance Traveled Example

Recall that we started this excursion with the following question:

Given the table of velocities and times

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| speed (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

can we determine how far the car traveled?

# Our Distance Traveled Example (Cont.)

It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

# Our Distance Traveled Example (Cont.)

It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

That is,

| t | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| f(t) | 20 | 30 | 38 | 44 | 48 | 50 |

# Our Distance Traveled Example (Cont.)

It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

That is,

| t | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| f(t) | 20 | 30 | 38 | 44 | 48 | 50 |

This is the curve under which we've been attempting to estimate the area.

# Our Distance Traveled Example (Cont.)

It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

That is,

| t | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| f(t) | 20 | 30 | 38 | 44 | 48 | 50 |

This is the curve under which we've been attempting to estimate the area. Later, we'll be able to explicitly compute that the area under this curve–which represents the distance traveled over those ten seconds–is

$$\frac{1175}{3} = 391.\overline{6} \text{ feet}$$

# Our Distance Traveled Example (cont.)



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

With 5 equidistant points

5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

# Our Distance Traveled Example (Cont.)

With 5 equidistant points

- Our Left-Hand Sum estimated 360 feet,

# Our Distance Traveled Example (Cont.)



5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method

With 5 equidistant points

- Our Left-Hand Sum estimated 360 feet,
- Our Right-Hand Sum estimated 420 feet,

# Our Distance Traveled Example (cont.)



With 5 equidistant points

- Our Left-Hand Sum estimated 360 feet,
- Our Right-Hand Sum estimated 420 feet,
- Our average estimated 390 feet, which was quite close.

# Our Distance Traveled Example (Cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|

# Our Distance Traveled Example (Cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |

# Our Distance Traveled Example (cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |
| 100 | 390.1625 |

# Our Distance Traveled Example (cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |
| 100 | 390.1625 |
| 1,000 | 391.516625 |

# Our Distance Traveled Example (cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |
| 100 | 390.1625 |
| 1,000 | 391.516625 |
| 10,000 | 391.65166625 |

# Our Distance Traveled Example (cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |
| 100 | 390.1625 |
| 1,000 | 391.516625 |
| 10,000 | 391.65166625 |
| 100,000 | 391.6651666625 |

# Our Distance Traveled Example (Cont.)

Here is a table of Left-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=0}^{n-1} f(t_i)\Delta t$ |
|---|---|
| 10 | 376.25 |
| 100 | 390.1625 |
| 1,000 | 391.516625 |
| 10,000 | 391.65166625 |
| 100,000 | 391.6651666625 |

So we can see that as $n$ increases, the Left-Hand Sums increase towards the actual area under the curve, as expected.

Math 122

Clifton

5.1: Distance and Accumulated Change

Constant Functions

Linear Functions

Non-Linear Functions

Right Endpoint Estimates

Left Endpoint Estimates

Partitions

Left- and Right-Hand Sums

Applying Our Method


Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|

# Our Distance Traveled Example (Cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |

# Our Distance Traveled Example (cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |
| 100 | 393.1625 |

# Our Distance Traveled Example (Cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |
| 100 | 393.1625 |
| 1,000 | 391.816625 |

# Our Distance Traveled Example (cont.)



Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |
| 100 | 393.1625 |
| 1,000 | 391.816625 |
| 10,000 | 391.68166625 |

# Our Distance Traveled Example (cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |
| 100 | 393.1625 |
| 1,000 | 391.816625 |
| 10,000 | 391.68166625 |
| 100,000 | 391.6681666625 |

# Our Distance Traveled Example (Cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

| $n$ | $\sum_{i=1}^{n} f(t_i)\Delta t$ |
|---|---|
| 10 | 406.25 |
| 100 | 393.1625 |
| 1,000 | 391.816625 |
| 10,000 | 391.68166625 |
| 100,000 | 391.6681666625 |

So we can see that as $n$ increases, the Right-Hand Sums decrease towards the actual area under the curve, as expected.