Math 122

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Calculus for Business Administration and Social Sciences

Outline

1. 4.1: Local Maxima and Minima

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Definition 1

Let p be a point in the domain of f and let (a, b) be an interval containing p.

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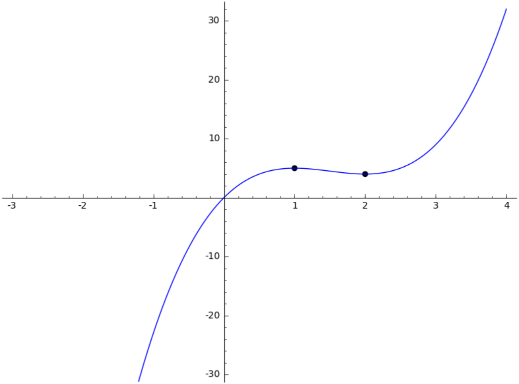
- If $f(x) \leq f(p)$ for every x satisfying $a < x < b$, then p is a *local maximum*.

Example



The point on the left is a local minimum, and the point on the right is a local maximum.

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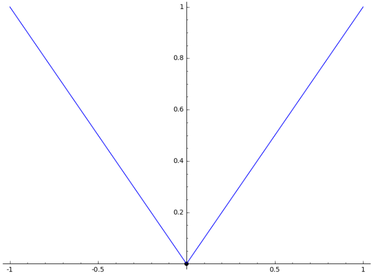

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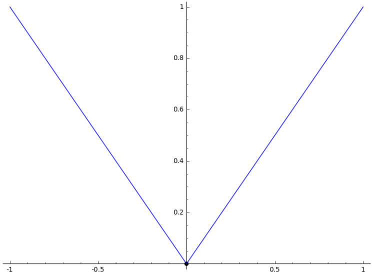

But the derivative is undefined at 0.

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A critical value of f is the function value, $f(p)$, at a critical point, p.

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Remark 1

The converse is **FALSE**. The point $x = 0$ is a critical point of $f(x) = x^2$, but neither a local minimum nor a local maximum.

Let f be a continuous function and let p be a critical point of f.

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- If $f''(p) = 0$, then the test gives no information.

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- The vertex is a local minimum if $0 < a$.

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- $f''(1) = 6(2(1) - 3) < 0$
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- $(1, 5)$ is a local maximum and $(2, 4)$ is a local minimum.

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- If f' is differentiable on an interval containing p and $f''(p) = 0$ or f'' is undefined, then p is a possible inflection point.
- If the signs of $f''(x_1)$ and $f''(x_2)$ are different for two points $x_1 < p$ and $p < x_2$, then p is an inflection point.

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Therefore $x = 3$ is an inflection point of f.

Example

The point $x = 0$ is a root of the second derivative of $f(x) = x^4$, but it is **not** an inflection point because

$$f''(x) = 12x^2$$

never changes sign.