Math 122

Ann Clifton [1]

[1] University of South Carolina, Columbia, SC USA

Calculus for Business Administration and Social Sciences

Outline

1. 2.4: The Second Derivative
   - Concavity

Outline

Definition

Definition 1

If both f and f' are differentiable functions, the *second derivative*, f'', is the derivative of f':

$$f''(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right)$$

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Remark 1

The second derivative tells us the rate of change of the first derivative, or the *acceleration*.

EXAMPLE

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- The first derivative gives the velocity of the object at time t,
$$v(t) = s'(t) = -9.8t.$$
- The second derivative gives the rate of change of acceleration (due to gravity)
$$a = v'(t) = s''(t) = -9.8\frac{\text{m}}{\text{s}^2}.$$

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- This is all observable from the graph of $s(t)$, which is a downward facing parabola. This is an example of a *concave down* function.
- Clearly, an upward facing parabola should be a *concave up* function.

Formal Definition

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- If $f''(x) < 0$ on an interval, then $f'(x)$ is decreasing on that interval and $f(x)$ is *concave down* on that interval.

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- If $f'' > 0$ on an interval, then $f'(x)$ is increasing on that interval and $f(x)$ is *concave up* on that interval.

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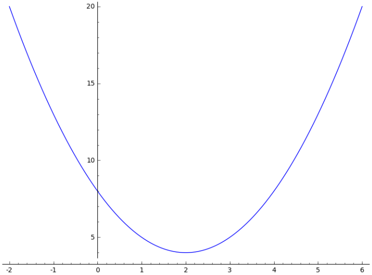

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- If f is continuous at every point in an interval (a, b), then we say that f is *continuous on* (a, b).
- If f is continuous at every point in its domain, then we simply say that f is *continuous*.

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- Logarithms.

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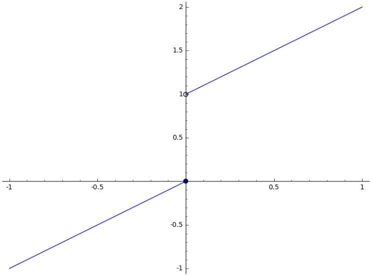

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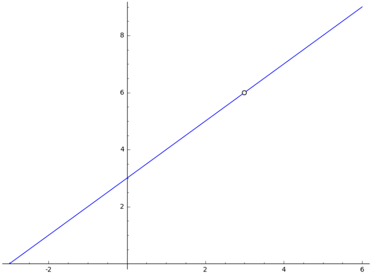

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