MATH 122

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Calculus for Business Administration and Social Sciences

Outline

Proportionality

Definition 1

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Remark 1

This is a fancy way of saying the function $y(x) = kx$ is a line through the origin.

Example

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(C) Estimate the heart mass of a horse with body mass of 650 kg.

Example (Cont.)

$$H(650) \approx 0.006(650)$$

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$$H(650) \approx 0.006(650) = 3.9 \text{ kg.}$$

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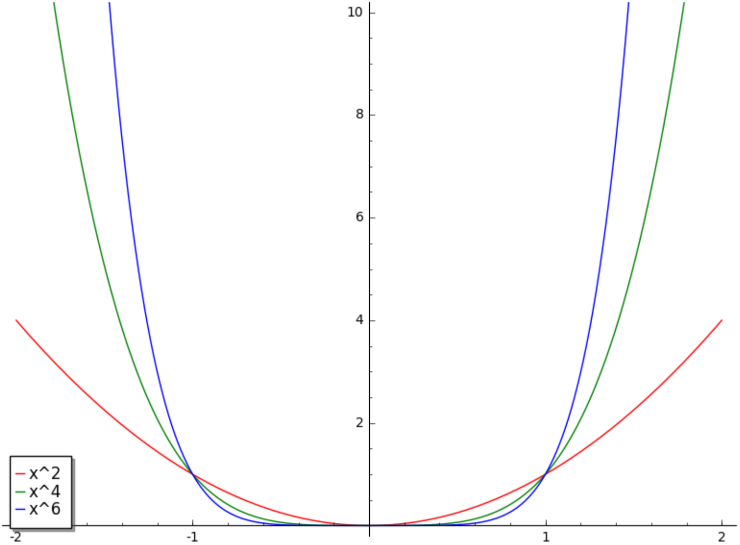
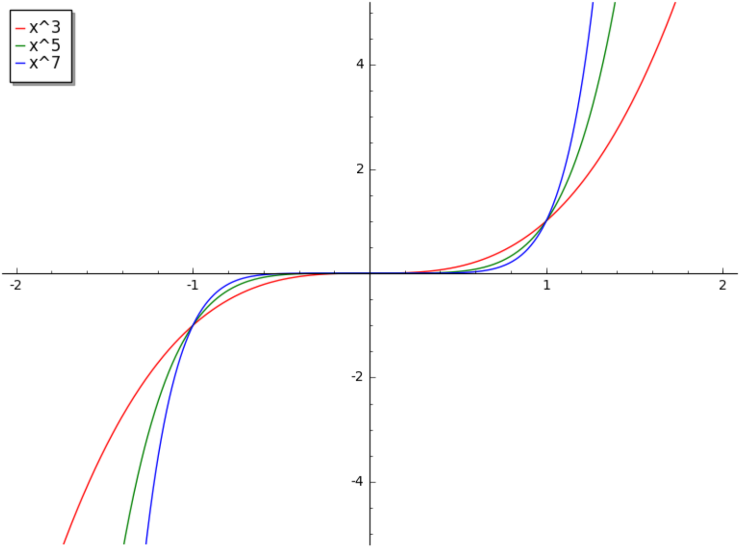
Remark 2

Generally, one calls this a *monomial* when $0 \leq p$.

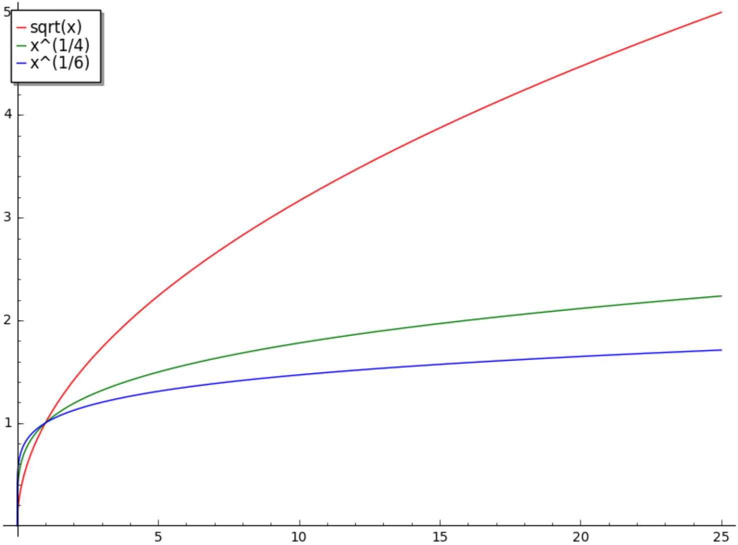
x^{2n}



x^{2n+1}



$\sqrt[2n]{x}$



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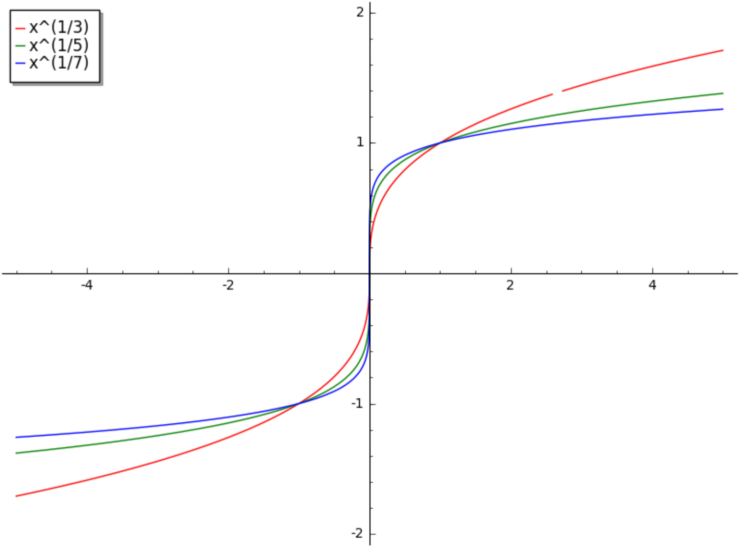
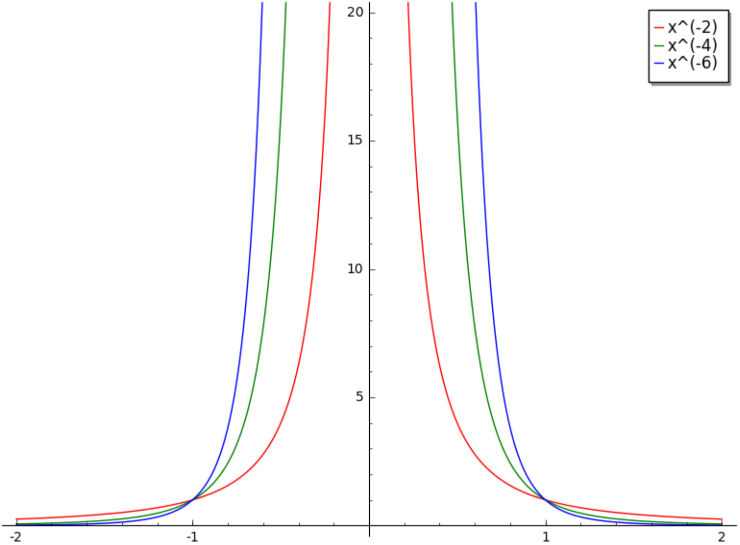
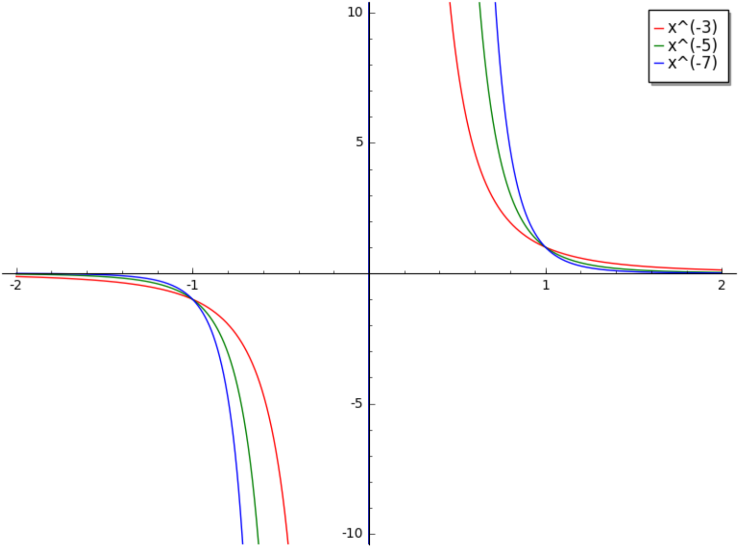


x^{-2n}



$x^{-(2n+1)}$



Polynomials

Definition 3

Sums of power functions with non-negative integer exponents

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \, a_n \neq 0$$

are *polynomials of degree n*.

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Remark 3

When the $n = 2$, one calls the polynomial a *quadratic*

Vertex Form

Any quadratic, $f(x) = ax^2 + bx + c$, can be written in vertex form

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the vertex of the parabola.

Vertex Form (Cont.)

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

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$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c
\end{aligned}
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&= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + \frac{4ac}{4a}
\end{aligned}$$

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&= a\left(x-\frac{-b}{2a}\right)^2+\frac{4ac-b^2}{4a}
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\end{aligned}$$

So $h=-b/2a$ and $k=f(h)=(4ac-b^2)/4a$.

The Quadratic Formula

The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. Put the quadratic in vertex form,

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3. Shift horizontally by h.
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5. Translate vertically by k.

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- At a price of $35 per person, the average number of people attending is 120.

Determine the price that will generate the greatest revenue assuming the number of people attending a concert is a linear function of the price.

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$$R(p) = p \cdot q(p) = p(-3p + 225) = -3p^2 + 225p.$$

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$$\begin{aligned} R(37.5) &= -3(37.5)(37.5 - 75) \\ &= -3(37.5)(-37.5) \\ &= 3(37.5)^2 \\ &= 4,218.75. \end{aligned}$$

Example (cont.)

It's clear from the graph that this is the maximum revenue: