MATH 122

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Calculus for Business Administration and Social Sciences

Outline

1. 1.8: New Functions from Old
   - Function Composition
   - Scaling
   - Rigid Transformations

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Function Composition

Definition 1

Given a function f and a function g such that the range of f is contained in the domain of g we can define the composition

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then $g(x) \neq 0$. Hence

$$f \circ g(x) = \frac{1}{x - 1}.$$

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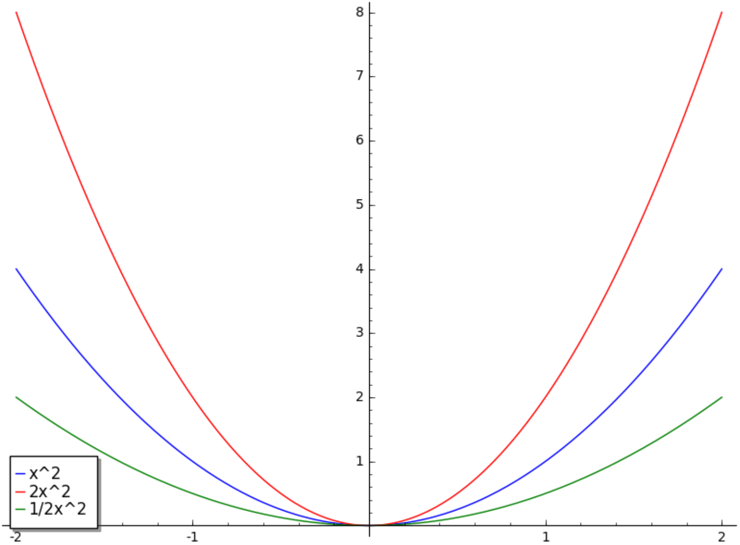
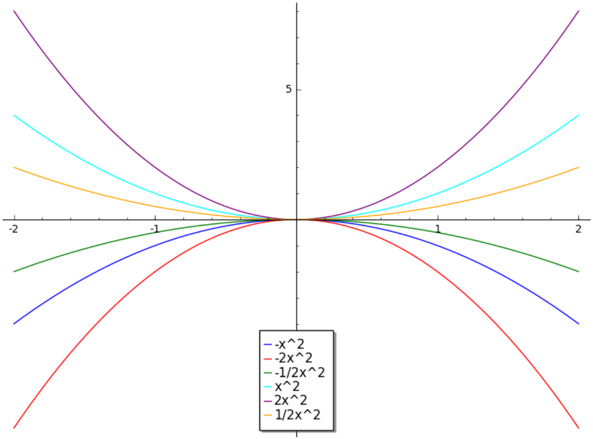
- a *vertical shrinking* of the graph of $f(x)$ if $a < 1$.

Examples



Reflection

The graph of $-f(x)$ is a reflection of $f(x)$ across the x-axis.



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- The graph of $f(x) - a$ is the graph of $f(x)$ shifted down a units.

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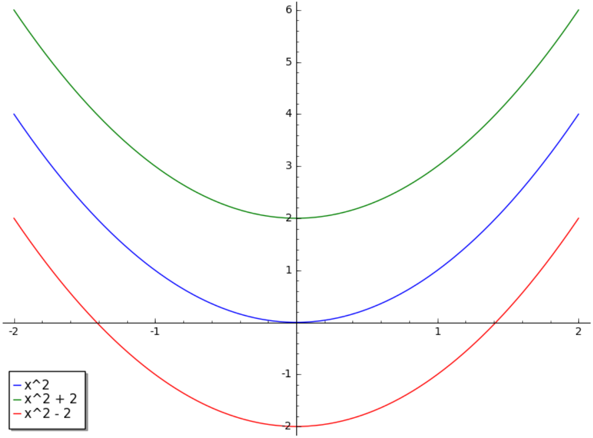

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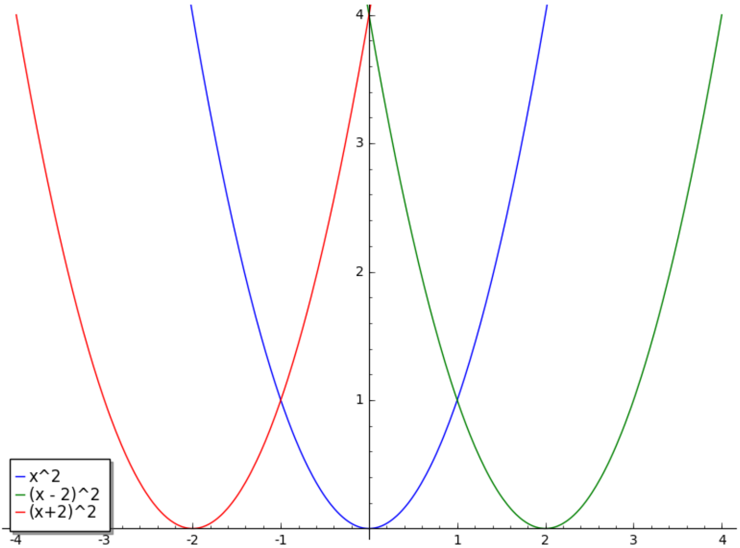
Examples



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- The *half-life* of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

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Computing Doubling Time/Half-Life

To approximate the value of the doubling time with a calculator:

$$d = log_a(2) = \frac{\ln(2)}{\ln(a)}$$

and

$$h = -\log_a(2) = -\frac{\ln(2)}{\ln(a)}.$$

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(B) How long will it take for the radiation levels to decay to the maximum acceptable radiation level of 0.6 millirems/hour set by the EPA?

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Therefore, it will take approximately $346.57/24 = 14.4$ days.

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Therefore

$$P(t) \approx 19.5e^{0.28t}.$$

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So at the end of the year, the balance will be $P_0\left(1+\frac{r}{n}\right)^n$.

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So at the end of the year, the balance will be $P_0\left(1+\frac{r}{n}\right)^n$. Continuing this way, the account balance after t years will be

$$P_0\left(1+\frac{r}{n}\right)^{nt}.$$

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Continuously Compounding Interest

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$$P(t) = P_0 e^{rt}.$$

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