MATH 122

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Calculus for Business Administration and Social Sciences

Outline

1 1.4: Applications of Functions to Economics

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Cost and Revenue

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- the profit from selling q goods by $\pi(q)$.

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- The $\$24,000$ expenditue is called a *fixed cost*.
- The \$7/radio in labour and material is called a *variable cost*.

Linear Marginal Cost

Definition 1

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Remark 1

This is just the slope of the linear cost function.

Profit

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- Given a revenue and a cost function, the profit function is

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- Given a revenue and a cost function, the profit function is
$$\pi(q) = R(q) - C(q).$$
- The *break-even* point is the quantity, q, for which
$$\pi(q) = 0$$
holds.

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hold. Therefore the break-even point is

$$q = \frac{24000}{8} = 3000.$$

Marginal Revenue

Definition 3

The *marginal revenue* for a linear revenue function is the revenue from selling one additional item,

$$\frac{R(q+1)-R(q)}{(q+1)-q} = R(q+1)-R(q).$$

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Remark 2

This is just the slope of the revenue function.

Marginal Profit

Definition 4

The *marginal profit* for linear cost and revenue functions is the profit from selling one additional item

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$$\frac{\pi(q+1)-\pi(q)}{(q+1)-q} = \pi(q+1)-\pi(q).$$

Remark 3

This is the slope of the revenue function less the slope of the cost function.

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- The base a is sometimes called the *growth/decay factor*.

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Exponential functions have constant **relative** change.
Linear functions have constant **rate** of change.

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$$\begin{aligned} P(17) &= P(0) \cdot a^{17} = 14a^{17} = 207 \\ \Rightarrow a^{17} &= \frac{207}{14} \\ \Rightarrow a &= \sqrt[17]{\frac{207}{14}} \approx 1.172 \end{aligned}$$

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Therefore,

$$P(t) = 14\left(\frac{207}{14}\right)^{\frac{t}{17}} \approx 14(1.172)^t.$$

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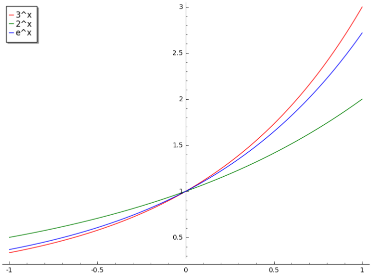
(B) Find the relative growth rate.

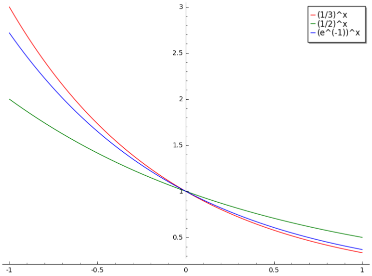
$$r = a - 1 = \sqrt[3]{\frac{91.4}{88.2}} - 1 \approx 0.012$$

Graphs of Exponential Functions

$1 < a$:



$0 < a < 1$:



Definition

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A function $f(x)$ has an *inverse* if there exists a function $f^{-1}(x)$ such that

$$f \circ f^{-1}(x) = x \text{ and } f^{-1} \circ f(x) = x.$$

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Theorem 1 (Horizontal Line Test)

If any horizontal line intersects the graph of $f(x)$ in **at most one** *point, then $f(x)$ admits a composition inverse.*

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- By definition,

$$\log_a(a^x) = x \text{ and } a^{\log_a(x)} = x.$$

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The *logarithm with base a* is the inverse function of the exponential function, a^x, and is denoted by

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- By definition,

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- One denotes $\log_e(x)$ by $\ln(x)$.

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- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$.

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\begin{aligned}
\Rightarrow \ln(3^t) &= \ln(10) \\
\Rightarrow t\ln(3) &= \ln(10) \\
\Rightarrow t &= \frac{\ln(10)}{\ln(3)} (= \log_3(10))
\end{aligned}
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$$\begin{aligned} \Rightarrow e^{3t} &= \frac{12}{5} \\ \Rightarrow \ln(e^{3t}) &= 3t = \ln\left(\frac{12}{5}\right) \\ \Rightarrow t &= \frac{1}{3}\ln\left(\frac{12}{5}\right) \end{aligned}$$

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$$P_0 e^{kt} = P_0 \left(e^k\right)^t$$

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We call k the *continuous growth/decay rate*.

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Let $a = e^{0.05}$. Then

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$$P(t) = 500(1.06)^t = 500e^{\ln(1.06)t}.$$