Lectures - Week 15
Line Integrals, Green's Theorems
and a Brief Look at Partial Differential Equations

Line Integrals

Another type of integral that one encounters in higher dimensions is a *line integral* where we want to integrate a quantity along a given curve. For example, we might want to determine the mass of a thin semi-circle shaped wire with variable density or to compute the work done in moving an object along a curved path; recall that if we move an object with a force $\vec{F}$ along the vector $\vec{PQ}$ from point P to Q (i.e., along a straight line) then the work done is $\vec{F} \cdot \vec{PQ}$. One application of the use of a line integral is the analogue of this formula if we move the object along a curve C.

We write the line integral of a *scalar function* (i.e., not a vector function) $f(\vec{x})$ along the given curve C as

$$\int_C f(\vec{x})\, ds$$

Here ds is the infinitesimal increment along the curve; you probably saw this in calculus when calculating arc length which was just $\int ds$ where in R^2

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This integral is typically evaluated by parameterizing the curve in terms of some parameter, say $x = g(t), y = h(t)$ to give

$$\int_C f(\vec{x})\, ds = \int_{t_0}^{t_1} f\big(g(t), h(t)\big) \left[\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right]\, dt$$

Another equivalent way to write this integral is using the parameterization of the curve $\vec{r} = g(t)\hat{\imath} + h(t)\hat{\jmath}$ (in R^2)

$$\int_C f(\vec{x})\, ds = \int_{t_0}^{t_1} f\big(\vec{r}(t)\big) \|\vec{r'}(t)\|_2\, dt$$

because $\vec{r'}(t) = (dx/dt)\hat{\imath} + (dy/dt)\hat{\jmath}$ so that $\|r'(t)\|_2 = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. The following example illustrates evaluating a line integral by both of the formulas.

Example Evaluate the line integral of $f(x,y) = xy^2$ along the curve defined by the portion of the circle of radius 2 in the right half plane oriented in a counterclockwise direction; i.e., along the circle $x^2 + y^2 = 4$ starting from $(0,-2)$ and ending at $(0,2)$.

The first thing we do is parameterize the curve in terms of a single variable, say t (i.e., we do a change of variables as we did before) so we let

$$x = 2\cos t \qquad y = 2\sin t \Rightarrow \frac{dx}{dt} = -2\sin t, \quad \frac{dy}{dt} = 2\cos t$$

We chose this parameterization because we then have $x^2+y^2 = 4\cos^2 t+4\sin^2 t = 4$ which is just the equation of the circle. Now we have to determine the starting and ending value of the parameter t. At the starting point $x = 0, y = -2$ so $t = -\pi/2$; at the ending point $x = 0, y = 2$ so $t = \pi/2$. Our line integral then becomes

$$\begin{aligned}\int_C xy^2\, ds &= \int_{-\pi/2}^{\pi/2} (2\cos t)(2\sin t)^2 \sqrt{4\sin^2 t + 4\cos^2 t}\, dt \\ &= \int_{-\pi/2}^{\pi/2} (4\cos t \sin^2 t)(2)\, dt = 8\int_{-\pi/2}^{\pi/2} \cos t \sin^2 t\, dt\end{aligned}$$

This integral is easily evaluated using the substitution $u = \sin t$, $du = \cos t dt$ to obtain

$$8\int_{-\pi/2}^{\pi/2} \cos t\sin^2 t\, dt = 8\int u^2\, du = \frac{8}{3}(\sin t)^3\Big|_{-\pi/2}^{\pi/2} = \frac{8}{3}(2) = \frac{16}{3}$$

Using the second formula we have $\vec{r} = 2\cos t\hat{\imath} + 2\sin t\hat{\jmath}$ so that $\vec{r'}(t) = -2\sin t\hat{\imath} + 2\cos t\hat{\jmath}$ and $\|\vec{r'}(t)\|_2 = 2$. Thus

$$\int_{t_0}^{t_1} f\big(\vec{r}(t)\big)\|\vec{r'}(t)\|_2\, dt = \int_{-\pi/2}^{\pi/2} (2\cos t)(2\sin t)^2(2)$$

which is the same that we obtained above.

Example Find the mass of a thin wire with density $\rho(x, y) = 2x - y + 1$ assuming the wire is in the shape of the unit circle.

We parameterize the curve as $\vec{r} = \cos t\hat{\imath} + \sin t\hat{\jmath}$ and thus $\vec{r'}(t) = -\sin t\hat{\imath} + \cos t\hat{\jmath}$ and $\|\vec{r'}(t)\|_2 = 1$. The limits of integration are $t = 0$ to $t = 2\pi$. To evaluate the mass we compute

$$\begin{aligned}\int_C \rho(x, y)\, ds &= \int_0^{\pi} \rho(\vec{r})\|\vec{r'}(t)\|_2\, dt = \int_0^{2\pi} (2\cos t - \sin t + 1)(1)\, dt \\ &= \big[2\sin t + \cos t + t\Big|_0^{2\pi} = (0 + 1 + 2\pi) - (0 + 1 + 0) = 2\pi\end{aligned}$$

We now want to use the line integral to compute the work done by moving an object along a curve C by a force field $\vec{F}$. In this case we are taking the line integral of a *vector function*. Again we parameterize the curve C by $\vec{r}(t)$; the work done by the force $\vec{F}$ along the curve C becomes

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r'}(t)\, dt$$

Example Find the work done in applying a force $\vec{F} = (y^2 - z^2)\hat{\imath} + 2yz\hat{\jmath} - x^2\hat{k}$ with the curve parametrically defined by

$$x = t^2, \quad y = 2t, \quad z = t \qquad 0 \le t \le 1$$

We have $\vec{r} = t^2\hat{\imath} + 2t\hat{\jmath} + t\hat{k}$ so $\vec{r'}(t) = 2t\hat{\imath} + 2\hat{\jmath} + \hat{k}$. In terms of the parameter t the force becomes $\vec{F} = (4t^2 - t^2)\hat{\imath} + 4t^2\hat{\jmath} - t^4\hat{k}$. Consequently

$$\begin{aligned}\int_C \vec{F} \cdot \vec{r'}(t)\, dt &= \int_0^1 \left[3t^2\hat{\imath} + 4t^2\hat{\jmath} - t^4\hat{k}\right] \cdot \left[2t\hat{\imath} + 2\hat{\jmath} + \hat{k}\right] dt \\ &= \int_0^1 (6t^3 + 8t^2 - t^4) dt = \frac{119}{30}\end{aligned}$$

Conservative forces and Independence of path

In some cases the work done does not depend on the path chosen; this means that $\int_{C_1} \vec{F} \cdot \vec{dr} = \int_{C_2} \vec{F} \cdot \vec{dr}$ for certain functions $\vec{F}$. In this case we say that the line integral is *independent of path*. When does this happen? Recall that for a function of a single variable, $f(x)$ the Fundamental Theorem of Calculus tells us that

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

An analogous result for line integrals would evaluate the function at the endpoints of the curve and thus it would be independent of the curve from point P to Q. So what we want is the force $\vec{F}$ to be written as the gradient of some scalar function ϕ; i.e., there exists ϕ such that $\vec{F} = \nabla\phi$. We call ϕ the *potential* and say that $\vec{F}$ is a *conservative force*. When $\vec{F} = \nabla\phi$ and the starting point of the curve C is denoted by P and the ending point by Q, we have the Fundamental Theorem for line integrals:

$$\int_C \vec{F} \cdot \vec{dr} = \int_C \nabla\phi \cdot \vec{r'(t)}\, dr = \phi(Q) - \phi(P)$$

Why does this work? Because

$$\begin{aligned}\int_C \nabla\phi \cdot \vec{r'(t)}\, dr &= \int_C \Big(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\Big) \cdot \Big(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\Big)\, dt \\ &= \int_C \Big[\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial t}\Big]\, dt = \int_C \frac{d\phi}{dt}\, dt = \phi(Q) - \phi(P)\end{aligned}$$

where we have simply applied the Fundamental Theorem of Calculus for the last step.

Example Calculate $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = (e^x \sin y - y)\hat{\imath} + (e^x \cos y - x - 2)\hat{\jmath}$ along any curve C starting at the point P and ending at the point Q. We first check to see that the work done is independent of path, i.e., if $\vec{F}$ is conservative which means if there is a ϕ such that $\nabla\phi = \vec{F}$; otherwise we can't do this problem because the path is not given. To this end we see that ϕ_x and ϕ_y must satisfy

$$\phi_x = e^x \sin y - y \qquad \phi_y = e^x \cos y - x - 2$$

Integrating, these equations we have

$$\phi(x,y) = e^x \sin y - yx + c_1(y) \qquad \phi(x,y) = e^x \sin y - xy - 2y + c_2(x)$$

Combining these we see that

$$\phi(x,y) = e^x \sin y - xy - 2y$$

and the force is conservative so we can just evaluate the integral at the endpoints of the curve; if the curve is closed, then clearly the result is zero. We have

$$\int_C \vec{F} \cdot \vec{dr} = e^x \sin y - xy - 2y\Big|_P^Q$$

Is there an easy check to see if a force is conservative? We can quickly check by the following

$$\text{if} \quad \vec{F} = M\hat{\imath} + N\hat{\jmath}, \quad \text{then } \vec{F} = \nabla\phi \text{ provided} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Note that this says that curl $\vec{F} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 0$. This works in three dimensions also; we check to see if curl$\vec{F} = 0$.

Example Verify that $\vec{F} = (-\pi \sin \pi x - yz)\hat{\imath} + (\pi \cos \pi y - xz)\hat{\jmath} - xy\hat{k}$ is conservative and find ϕ such that $\nabla\phi = \vec{F}$.
We compute the curl of $\vec{F}$ as

$$\det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\pi \sin \pi x - yz & \pi \cos \pi x - xz & -xy \end{pmatrix}$$
$$= \hat{\imath}(-x + x) + \hat{\jmath}(-y + y) + \hat{k}(-\pi \sin \pi x - z + \pi \sin \pi x + z) = \vec{0}$$

To find ϕ such that $\vec{F} = \nabla\phi$ we note that

$$\phi_x = -\pi \sin \pi x - yz \quad \phi_y = \pi \cos \pi y - xz \quad \phi_z = -xy$$

which implies

$$\phi = \cos \pi x - xyz + C_1(y,z) \qquad \phi = \sin \pi y - xyz + C_2(x,z) \qquad \phi = -xyz + C_3(x,y)$$

and thus $\phi = \cos \pi x + \sin \pi y - xyz$.

Greens Identitites

Greens theorem and identities in R^2 relate a double integral over a region in the plane to a line integral over a closed curve in the region; the result can be extended to a triple integral and the integral over a surface. The result is stated formally in the following theorem but it is more widely known in various alternate forms which we will present.

Theorem Let D be a simply connected region with positively oriented piecewise smooth closed curve. Let $\vec{G} = M\hat{\imath} + N\hat{\jmath}$ be continuously differentiable. Then

$$\oint M\ dx + N dy = \int\int_D \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\ dA$$

One thing this tells us is that we have a choice of evaluating a line integral or a double integral; sometimes one is much easier than the other as the following example illustrates.

Example Evaluate $\oint (-y)\ dx + x\ dy$ along the path of the unit circle in the upper half plane starting at $(1, 0)$ going counterclockwise and returning to $(0, 1)$ via the x axis by transforming to a double integral.

By Greens Theorem with $M = -y$, $N = x$, $M_y = -1$, $N_x = 1$ we have

$$\oint (-y)\ dx + x\ dy = \int\int_D (1+1)\ dxdy = 2\frac{\pi(1)^2}{2} = \pi$$

because D is just the semicircle with area $.5\pi$.

There are three alternate forms of this result that we will look at; these are well known results in vector calculus. We will state them in R^2 for simplicity.

1. The Divergence (or Gauss) Theorem

$$\oint \vec{F}\cdot\vec{n}\ ds = \int\int_D \nabla\cdot\vec{F}\ dxdy$$

2. Stokes Theorem

$$\oint \vec{F}\cdot\vec{dr} = \int\int_D \mathrm{curl}\vec{F}\cdot\vec{n}\ dxdy$$

   where $\vec{n}$ is the unit outer normal to the curve.

3. $$\oint \frac{\partial\phi}{\partial\vec{n}}\ ds = \oint \nabla\phi\cdot\vec{n}\ ds = \int\int \nabla\cdot\nabla\phi\ dxdy = \int\int \Delta\phi\ dxdy$$

   Here $\frac{\partial\phi}{\partial\vec{n}}$ is the common notation used for the direction derivative of ϕ in the direction of the unit outer normal, i.e., $\nabla\phi\cdot\vec{n}$.

To see how we might derive these results from Green's Theorem we consider the Divergence Theorem. We take a function $\vec{F} = N\hat{\imath} - M\hat{\jmath}$. Then the divergence of $\vec{F}$ is $N_x - M_y$ which is just the right hand side of Green's Theorem. For the line integral $\oint \vec{F}\cdot\hat{n}\ ds$ we take the unit outer normal to be $dy\hat{\imath} - dx\hat{\jmath}$ because the vector $dx\hat{\imath} + dy\hat{\jmath}$ is the tangential vector. Then for $\vec{F} = N\hat{\imath} - M\hat{\jmath}$, we have $\oint \vec{F}\cdot\hat{n} = \oint N dy + M dx$ which is just the line integral in Green's Theorem.

Greens identities play the role of integration by parts in higher dimensions.

- Greens First Identity

$$\int\int_D f\Delta g + \int\int_D \nabla f \cdot \nabla g = \oint f \frac{\partial g}{\partial n}\, ds$$

- Greens Second Identity

$$\int\int_D \Big(f\Delta g - g\Delta f \Big) = \oint \Big(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \Big)\, ds$$

The proof of the first identity follows from applying the Divergence Theorem to $f(\nabla g)$ to obtain

$$\int\int \nabla \cdot (f\nabla g)\, dA = \oint f\nabla g \cdot \vec{n}\, ds = \oint f \frac{\partial g}{\partial \vec{n}}$$

We now use properties of ∇ to simplify the left hand side to get

$$\int\int \nabla \cdot (f\nabla g)\, dA = \int\int \Big[\nabla f \cdot \nabla g + f\nabla \cdot \nabla g \Big]\, dA = \int\int \Big[\nabla f \cdot \nabla g + f\Delta g \Big]\, dA$$

and the result follows by combining these two equations. The proof of the second Greens Identity is a straightforward application of the first identity with the roles of f and g reversed.

A Brief Introduction to Partial Differential Equations (PDEs)

We can take what we have learned from solving a BVP of one independent variable to solving a BVP of two independent variables. For example, suppose we want to find $u(x, y)$ which satisfies

$$-(u_{xx} + u_{yy}) = -\Delta u = f(x, y) \quad 0 < x < 1, \quad 0 < y < 1$$

$$u = g \quad \text{on the boundary}$$

This problem is called Laplace's equation if $f = 0$ and Poisson's equation otherwise.
In this case we have to discretize our domain in both x and y. We have

$$x_0 = 0, \quad x_i = x_{i-1} + \Delta x, \quad x_{n+1} = 1$$

$$y_0 = 0, \quad y_i = y_{i-1} + \Delta y, \quad y_{m+1} = 1$$

Then we let U_{ij} denote our discrete solution at (x_i, y_j). Then we can simply replace each second derivative with the second centered difference quotient. Writing our equation at (x_i, y_j) gives

$$\frac{-U_{i+1,j} + 2U_{ij} - U_{i-1,j}}{\Delta x^2} + \frac{-U_{i,j+1} + 2U_{ij} - U_{i,j-1}}{\Delta y^2} = f(x_i, y_j)$$

Once again we have a linear system to solve for our U_{ij}, $i = 1, n$, $j = 1, m$, i.e., we now have nm unknowns so if the number of unknowns in the x and y directions is the same, we have n^2 unknowns compared with n unknowns in 1-d. Clearly in R^3, we would have n^3 unknowns. This adversely affects the amount of work to solve the system because recall that the work was quantified in terms of the size of the matrix to some power. Also note that in this case we do not get a tridiagonal system. Actually what we get is a banded matrix, i.e., $A_{ij} = 0$ for $|i - j| > q$ which can also be solve efficiently. In one spatial dimension, i.e., for a two point BVP (an ODE), we obtained a tridiagonal system of linear equations to solve.

We can have other boundary conditions besides specifying $u = g$ on the boundary. We can also specify the derivative in the direction of the unit outer normal, i.e., $\partial u/\partial \vec{n} = \nabla u \cdot \vec{n}$. However, we can NOT specify this derivative boundary condition on all of the boundary. Why? When we specify $u = g$ on the boundary we call this a Dirichlet boundary condition and when we specify $\partial u/\partial \vec{n} = g$ on the boundary we call this a Neumann boundary condition. We can also have a mixed (or Robin) boundary condition $\alpha u + \beta \partial u/\partial \vec{n} = g$.

We can combine the work we did for IVPs and BVPs to solve the problem of finding $u(x, t)$ satisfying

$$u_t - u_{xx} = f(x, t) \quad x_L < x < x_R, \quad t > 0$$

$$u(x, 0) = u_0$$

$$u(x_L, t) = \alpha(t) \qquad u(x_R, t) = \beta(t)$$

Note that we specify u on the boundaries and u initially. This problem is called an initial boundary value problem (IBVP) and is known as the one-dimensional heat or diffusion equation. Now we let U_i^n be our approximation to the solution at time t^n and spatial point x_i. For example, if we write the difference equation at (x_i, t^n) and use a Forward Euler in time and our second centered difference in space we obtain

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} = f(x_i, t^n)$$

Note that in this case we have an explicit method and we can solve directly for U_i^{n+1} for all $i = 1, \ldots, n$. However, these explicit methods have severe time step restrictions and so one usually has to take a lot of small time steps which can lead to roundoff error accumulating. If, instead, we use a Backward Euler scheme sitting at point (x_i, t^{n+1}) we have the difference equation

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} = f(x_i, t^n)$$

In this case, at time t^{n+1} all the U_i are coupled and so we have to solve a linear system at each time step. If our original PDE is nonlinear, then we need to solve a nonlinear system at each time step, i.e., use Newton's iteration which requires solving several linear systems at each time step. However, we have a good initial guess for Newton's method at each time step. What is it?

The two examples of PDEs we have looked at are both second order linear PDEs. PDEs are classified in the same way as ODEs; e.g., by order, by linearity, by homogeneity, and whether a system or a single equation. A general second order PDE in two variables ξ, η is given by

$$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = g(\xi, \eta)$$

For example, for Laplace's equation $\xi = x$, $\eta = y$, $a = -1$, $b = 0$, $c = -1$ and $d = e = f = 0$. For the heat equation, $\xi = x$, $\eta = t$, $a = -1$, $b = 0$, $c = 0$, $d = 0$, $e = 1$, $f = 0$. Second order linear PDEs are classified by the sign of the discriminant $b^2 - 4ac$. If

$$b^2 - 4ac = 0 \quad \text{the equation is parabolic,}$$

$$b^2 - 4ac < 0 \quad \text{the equation is elliptic}$$

$$b^2 - 4ac > 0 \quad \text{the equation is hyperbolic}$$

Note that for Laplace's equation $b^2 - 4ac = 0 - 4 < 0$ so it is an elliptic equation; Laplace's equation (or equivalently Poisson's equation) is the prototype elliptic equation and models a state of equilibrium because no time is involved. For the heat equation $b^2 - 4ac = 0 - 0 = 0$ so the heat equation is a parabolic equation and is the prototype parabolic equation which models the time dependent phenomena of diffusion. The wave equation $u_{tt} - u_{xx} = f(x,t)$ is the prototype equation for hyperbolic equations because $b^2 - 4ac = 1 - 4(0) = 1 > 0$ and models, e.g., the waves or oscillations. In general, hyperbolic equations are the most difficult to solve.

Because PDEs are so important in modeling physical phenomena, there are many methods that have been developed for approximating their solution. We have briefly looked at finite difference methods where the derivatives are replaced by difference quotients. Another common method is finite element methods where the solution of a related weak problem is approximated by piecewise polynomials. For example, if we have Poisson's equation

$$-\Delta u = f(x,y) \text{for } (x,y) \in \Omega \quad u = 0 \text{ on the boundary } \Gamma$$

then the weak formulation is found by multiplying the equation by an appropriate function v and integrating over the spatial domain and then applying Green's Theorem to balance the order of derivatives. We have

$$-\int_\Omega \Delta u v = \int_\Omega fv \Rightarrow \int_\Omega \nabla u \cdot \nabla v + \int_\Gamma \frac{\partial u}{\partial n} v = \int_\Omega fv$$

If we assume $v = 0$ on Γ then we have the weak problem of finding a u such that $u = 0$ on Γ and satisfing

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$

This is called a weak problem for our classical Poisson's equation because it admits solutions that the classical problem doesn't, i.e., it has weaker requirements on f and thus on u. However, if u satisfies the classical Poisson's equation then it satisfies the weak problem.

There are many other methods such as finite volume methods, spectral methods, level set methods, to name a few.