1. H. L.

Show: \( \triangle AZY \cong \triangle YCX \cong \triangle ZXB \cong \triangle XZY \).

Proof: \( Z = \text{midpt} \ (AB) \) \quad 1.31

\( ZY \parallel BC \) and \( ZY = \frac{1}{2} BC = BX = XC \)

Similarly, \( 1.31 \Rightarrow XY \parallel AB \) and \( XY = \frac{1}{2} AB, XZ \parallel AC \) and \( XZ = \frac{1}{2} AC \)

We have: \( ZY = BX = XC \)

\( \begin{align*}
AZ &= BZ = XZ \\
AY &= ZX = YZ
\end{align*} \)

\( \Rightarrow \triangle AZY \cong \triangle ZXB \cong \triangle YCX \)

We also have: \( XZ = AY \)

\( \begin{align*}
XY &= AZ \\
YZ &= YZ
\end{align*} \)

\( \Rightarrow \triangle AYZ \cong \triangle XZY \).
Show:WXYZ is a parallelogram.

Proof:

Apply 1.31 to $\triangle AOC$: $WZ \parallel AC$, $WZ = \frac{1}{2} AC$$\implies$$WZ \parallel XY$ and $WZ = XY$

$\implies$ WXYZ is a parallelogram.
Let \( \angle AOB = \angle ACB \).

Show: \( \angle ABO = \angle ACO \).

**Proof:** Consider \( \triangle OAX \sim \triangle CXB \)

\[ \angle AOX = \angle BCX \text{ (given)} \]
\[ \angle AOX = \angle BXC \text{ (vert. angle)} \]

Corr. \( \frac{XC}{XA} = \frac{XB}{XC} \quad \frac{XO}{XA} = \frac{XC}{XB} \) . Next, consider \( \triangle BCA \sim \triangle CDO \)

\[ \angle BCA = \angle OXC \text{ (vert. angle)} \]
\[ \frac{XC}{XB} = \frac{XA}{AC} \]

\( \Rightarrow \angle ABO = \angle ACO \).
Alternative proof using circles.

$S_1$: circumcircle of $\triangle ABO$  

In $S_1$, $\angle AOB = \frac{1}{2} AB = \angle AO_1 B$.

$\Rightarrow \angle AO_1 B = \angle AO_2 B$  \hspace{0.5cm} \text{To show: $\triangle A_1 O_2 B \cong \triangle A_0 O_2 B$.}$

In $\triangle A_0 O_2 B$, $A_0 = B_0$ \hspace{0.5cm} \angle O_1 AB = \angle O_2 BA \hspace{0.5cm} \angle O_2 AB = \angle O_2 BA \hspace{0.5cm} \angle O_2 AB = \angle O_2 AB$

In $\triangle A_0 O_2 B$, $\angle O_1 AB + \angle O_1 BA + \angle O_2 BA = \angle O_0 O_2 B + 2 \angle O_2 AB = 180^\circ$

In $\triangle A_0 O_2 B$, $\angle O_1 AB + \angle O_2 AB + \angle O_2 BA = \angle O_0 O_2 B + 2 \angle O_2 AB = 180^\circ$.

We now have: \[ \angle O_1 AB = \angle O_2 AB \]

We now have: \[ \angle AO_2 B = \angle AO_2 B \]

\[ \angle O_1 AB = \angle O_2 AB \]

\[ \angle AB = \angle AB \]

\[ \angle AO_2 B \cong \angle AO_2 B \]

\[ \triangle A_0 O_2 B \cong \triangle A_0 O_2 B \]

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\[ \triangle A_0 O_2 B \cong \triangle A_0 O_2 B \]
\[ \text{Consider } \overline{AO_2D}, \overline{AO_1D} \]
\[ \begin{align*}
AO_2 &= AO_1 \\
\angle O_2AD &= \angle O_1AD \\
\text{SAS} &\implies \triangle AO_2D \cong \triangle AO_1D \\
AD &= AD
\end{align*} \]

\[ \implies O_2 = O_1. \text{ We now have: } O_2 = O_1 = AO_1 = AO_2 \]

\[ A, O \text{ both on } S_2 \]

\[ \implies O \text{ is on the circle with radius } AO_2, \text{ namely } S_2. \]

\[ \implies A, B, C, D \text{ are on } S_2 \text{ (ABCD is cyclic).} \]

To conclude, we observe that \[ \angle ACO = \frac{1}{2} \overline{AO} \equiv \angle ABO. \]
11.5.

Suppose that LT is tangent to S.

Show: \((XY)(XT) = (LT)^2\).

Proof:

We will show that \(\triangle EYT \sim \triangle ETX\).

We observe: LT tangent to S

\[\Rightarrow \angle ETY = \frac{\pi}{2} \angle LT.\]

But also, we have

\[\angle EYT = \frac{\pi}{2} \angle LT, \text{ so } \angle ETY = \angle EXT.\]

Consider:

\[\triangle EYT \sim \triangle ETX\]

\[\angle E = \angle E, \text{ and } \angle ETY = \angle EXT\]

\[\therefore \triangle EYT \sim \triangle ETX\]

\[\Rightarrow \frac{LY}{LT} = \frac{LT}{LX} \Rightarrow (XY)(XT) = (LT)^2.\]
LH.C.

Suppose: \( XY \parallel AB \),
\( YZ \parallel BC \).

Show: \( XZ \parallel AC \).

**Proof:** Apply 1.29 in \( \triangle ABE \): \( XY \parallel AB \Rightarrow \frac{EX}{EA} = \frac{EY}{EB} \)

\[ \triangle BCE : YZ \parallel BC \Rightarrow \frac{EZ}{EC} = \frac{EY}{EB} \]

\[ \Rightarrow \frac{EX}{EA} = \frac{EZ}{EC} \]

1.29 \( \Rightarrow \) \( XZ \parallel AC \).

in \( \triangle ACE \)
14.7.

Suppose: \( PQ \parallel AQ, \)
\( QX \parallel BP. \)

Show: \( XY \parallel AB. \)

Proof:

Apply 1.29 to \( \triangle QPB: \) \( QX \parallel BP \Rightarrow \frac{CX}{CB} = \frac{CQ}{CQ} \times \frac{CL}{CB} \Rightarrow (CX)(CB) = (CL)(CQ) \)

\( \triangle CAQ: \) \( PQ \parallel AQ \Rightarrow \frac{CL}{CA} = \frac{CQ}{CQ} \times \frac{CA}{CQ} \Rightarrow (CL)(CQ) = (CA)(CQ) \)

\( \Rightarrow (CX)(CB) = (CA)(CQ) \Rightarrow \frac{CX}{CA} = \frac{CX}{CB} \Rightarrow \frac{CX}{CA} \times \frac{1}{CB} = XY \parallel AB. \)
Suppose: \( UV \parallel BC \)

Show: \( \frac{UV}{VV} = \frac{BX}{XC} \)

Proof: Consider

1. \( \triangle AUV \sim \triangle ABX \)
   \[ \angle UAV = \angle BAX \text{ (same angle)} \]
   \[ \angle AUV = \angle ABX \text{ (cor. angle)} \]
   \[ \implies \triangle AUV \sim \triangle ABX \]

   \[ \text{cor.} \]
   \[ \frac{AV}{AX} = \frac{UV}{BX} \]

2. \( \triangle AUV \sim \triangle AXC \)
   \[ \angle UAV = \angle XAC \text{ (same angle)} \]
   \[ \angle AUV = \angle AXC \text{ (cor. angle)} \]
   \[ \implies \triangle AUV \sim \triangle AXC \]

   \[ \text{cor.} \]
   \[ \frac{AX}{AX} = \frac{XX}{XC} \]

   \[ \implies \frac{UV}{BX} = \frac{VV}{XC} \]

   \[ \implies \frac{UV}{VV} = \frac{BX}{XC} \]
Suppose: \( \overline{SA} \) is tangent to \( S \), \( \overline{AM} = 1 \), and \( M = \text{midpt} \ (\overline{AB}) \) (so \( \overline{AM} = \overline{MB} \)).

Find \( \overline{AB} \).

Solution: Same reasoning as in 1.H.5. shows that \( \triangle AMR \sim \triangle LAB \).

\[
\frac{\overline{AB}}{\overline{AM}} = \frac{\overline{PB}}{\overline{LA}} = \frac{\overline{LM}}{\overline{LM}}, \quad \text{Note:} \quad \overline{AM} = \overline{MB} \implies \overline{LM} = \frac{1}{2} \overline{LB}.
\]

\[
\overline{PB} = \overline{LA} \times \frac{\overline{LA}}{\overline{LM}} \quad \overline{LM} \cdot \overline{PB} = (\overline{LA})^2 \iff \overline{LM} = \frac{1}{2} (\overline{LB})^2 = (\overline{LA})^2
\]

\[
\implies \overline{PB} = \sqrt{2} (\overline{LA}).
\]

\[
\overline{AB} = \overline{AB} = \frac{\overline{PB}}{\overline{AM}} = \frac{\sqrt{2} (\overline{LM})}{\overline{LA}} = \sqrt{2}. \quad \text{I.e.,} \quad \overline{AB} = \sqrt{2}
\]