Solutions 1E.

Problems 1, 3.

(1)

Suppose: \(AX = \text{med}(A)\)
\(BX = \text{med}(B)\)

Show: \(A(\pi_1) = A(\pi_3)\)
\(A(\pi_2) = A(\pi_4)\)

Proof: We first prove that \(A(\triangle ABX) = A(\triangle AXY)\).

We have: \(AX = \text{med}(A) \Rightarrow BX = XC \Rightarrow \frac{1}{2} BC = BX\).

It follows that
\[
A(\triangle ABX) = \frac{BX \cdot \text{alt}(A)}{2} = \frac{1}{2} BC \cdot \frac{\text{alt}(A)}{2} = \frac{1}{2} \left( \frac{BC \cdot \text{alt}(A)}{2} \right) = \frac{1}{2} A(\triangle ABC)
\]

Similarly, \(BY = \text{med}(B) \Rightarrow \frac{1}{2} AC = AY\)

\[
\Rightarrow A(\triangle AXY) = \frac{AY \cdot \text{alt}(B)}{2} = \frac{1}{2} \left( \frac{AC \cdot \text{alt}(B)}{2} \right) = \frac{1}{2} A(\triangle ABC)
\]

Comparing these yields \(A(\triangle ABX) = A(\triangle AXY)\).
Now, observe that
\[ A(\Delta ABX) = A(\pi_1) + A(\pi_2) \]
\[ A(\Delta ABX) = A(\pi_2) + A(\pi_4). \]

Since \( A(\Delta ABY) = A(\Delta ABX) \), we have
\[ A(\pi_1) + A(\pi_2) = A(\pi_2) + A(\pi_4) \implies A(\pi_1) = A(\pi_4). \]

To conclude, we require \( A(\Delta ABX) = A(\Delta ACX) \):
\[ A(\Delta ABX) = \frac{1}{a} BX \cdot \text{area}(A) = \frac{1}{2} XC \cdot \text{area}(A) = A(\Delta ACX) \]
\[ BX = XC. \]

Now, we note that \( A(\Delta ABX) = A(\pi_1) + A(\pi_4) \):
\[ A(\Delta ACX) = A(\pi_2) + A(\pi_3). \]

Since these quantities are equal, and since \( A(\pi_1) = A(\pi_4) \), we have \( A(\pi_1) + A(\pi_4) = A(\pi_2) + A(\pi_3) \implies A(\pi_2) = A(\pi_3). \]
Give a formula for \( A = A(\Delta ABC) \) using only \( a, b, C \).

**Solution:**

We know that \( A = \frac{1}{2} ab \sin C \)

Law of sines: \( \frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow b = \frac{a \sin B}{\sin A} \)

We also have: \( \angle A + \angle B + \angle C = 180^\circ \Rightarrow \angle A = 180^\circ - (\angle B + \angle C) \)

We now conclude: \[ A = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (180^\circ - (B+C))} \]

(Fact: \( \sin (180^\circ - \theta) = \sin \theta \) \( = \sin 180^\circ \cos (-\theta) + \cos 180^\circ \sin (-\theta) = \sin \theta \))

Hence, we have \( \sin (180^\circ - (B+C)) = \sin (B+C) \).

It follows that \[ A = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B+C)} \]