Solutions 10

Problems 4, 6, 7, 8, 9, 10; challenge probs: 12, 13.

(4) Let ABCD be a parallelogram. Suppose that the diagonals are perpendicular. Show: ABCD is a rhombus.

Proof: We suppose that ABCD is a parallelogram and that diagonals $AC \perp BD$. To show that ABCD is a rhombus, it suffices to show that the 4 sides are equal.

We have:

- ABCD: parallelogram $\Rightarrow$ $AX = XB$ (diags. bisect each other)
- $AC \perp BD \Rightarrow \angle AXD = \angle AXB = 90^\circ$.

We therefore consider:

$$\triangle AXD \cong \triangle AXB$$

$$AX = AX$$

$$\angle AXD = \angle AXB$$

$$DX = XB$$

$\Rightarrow$ AD = AB. We now have:

ABCD parallelogram $\Rightarrow \{ AB = CD \} \quad \{ \text{opp. sides} \}$

$\Rightarrow$ all 4 sides equal; $\Rightarrow$ ABCD is a rhombus.
Suppose that $AB \parallel CD$, but that $AD \neq BC$.

Show: $\angle D = \angle C \iff AD = BC$.

Proof: Draw $X$ on $DC$ with $AD \parallel BX$. Then we have $AB \times D$ is a parallelogram (opp sides parallel).

It follows that $AB = DX \parallel BC$ (opp sides equal).

$AD = BX$.

$(\implies)$ Suppose: $\angle D = \angle C$. To show: $AD = BC$.

To start, $AD \parallel BX \implies \angle L = \angle D$ (corr. angles).

We have:

$\begin{align*}
\angle D &= \angle C \\
\angle L &= \angle D
\end{align*}$

$\implies \angle L = \angle C \implies \Delta BXC$ is iso., hence $CX 

\implies BX = BC$. Observe: $AD = BX \parallel BC$.

$\implies AD = BC$.

$(\impliedby)$ Suppose: $AD = BC$. To show: $\angle D = \angle C$.

We have:

$\begin{align*}
AD &= BC \\
AD &= BX
\end{align*}$

$\implies BC = BX \implies \Delta BXC$ is iso., hence $CX$

$\implies \angle C = \angle L$. But also note that $\angle D = \angle L$ since $AD \parallel BX$ (corr. angles).

It follows that $\angle D = \angle L \implies \angle C = \angle D$. 
(7) Let \(ABCD\) be a parallelogram. Show that opposite interior angles are equal.

**Proof:**

Draw diagonals \(AC\) and \(BD\).

\(ABCD\) is a parallelogram \(\Rightarrow\) \(\angle 1 = \angle 3\) (alt. int. angles)

We now have \(\angle A = \angle 1 + \angle 2 = \angle 3 + \angle 4 = \angle BC\). I.e., we have \(\angle A = \angle BC\).

Similarly, one can show that \(\angle B = \angle D\).

(8) Let \(ABCD\) be a quadrilateral. Suppose that \(AB \parallel CD\) and \(\angle B = \angle D\). Show: \(ABCD\) is a parallelogram.

**Proof:**

Since \(AB \parallel CD\), it suffices to show that \(AD \parallel BC\).

We have: \(AB \parallel CD \Rightarrow \angle 1 = \angle 2\) (alt. int. angles)

It follows that \(\angle B = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle D\) (subtends)

and \(\angle 1 = \angle 2 \Rightarrow \angle 3 = \angle 4\)

\(\Rightarrow AD \parallel BC\) (alt. int. angles)
(9) Let $ABCD$ be a quadrilateral. Suppose that $\angle A = \angle C$ and $\angle B = \angle D$. Show: $ABCD$ is a parallelogram.

**Proof:**

We will show: $AB \parallel CO$ and $AD \parallel BC$.

**Sum of interior angles:**

$\angle A + \angle B + \angle C + \angle D = 2\angle A + 2\angle B = 360^\circ$

$\Rightarrow \angle A + \angle B = 180^\circ$.

We now have:

$\angle A + \angle B = 180^\circ \Rightarrow \angle B - \angle A = 0^\circ$

$\Rightarrow \angle B = \angle A$

$\Rightarrow EO = AD \parallel BC$ (alt. int. angles)

Similarly, one can show that $AB \parallel CO$.

$\Rightarrow ABCD$ is a parallelogram.
(10) Let \( ABCD \) be a nontrivial trapezoid. Show its diagonal is equal.

**Proof:**

Suppose: \( AB \parallel CO, AO = BC \).

To show: \( AC = BO \).

10.6: \( AO = BC \implies \angle DO = \angle OC \). Consider:

\[
\begin{align*}
\triangle ADO & \sim \triangle BCO \\
AO &= BC \\
\angle DO &= \angle OC \\
OC &= DC
\end{align*}
\]

\( \Delta ADO \sim \Delta BCO \)

\( \angle DAC = \angle BCD \)

\( \triangle ADO \sim \triangle BCO \)

\( AC = BO \).
Suppose: \(ABCD\) is a parallelogram, and \(AX = AY\).

Show: \(ABCO\) is a rhombus.

**Proof:** Since \(ABCD\) is a parallelogram,

\[
\begin{align*}
AB &= CO \quad \text{(opp. sides equal)} \\
AD &= BC
\end{align*}
\]

\(\angle B = \angle D\) (opp. angles equal by 10.9) be equal.

Consider: \(\triangle AXO \sim \triangle AYB\)

\[
\begin{align*}
\angle O &= \angle B \\
\angle AXO &= \angle AYB \quad (=90^\circ) \\
AX &= AY
\end{align*}
\]

\(\triangle AXO \cong \triangle AYB\) (AAS)

Corr. \(AB = AD\) sides.
Suppose: \( OA = OC = OB \), \( OT = TX \), \( OS = SY \), \( OR = RZ \).

Show: (1) \( \triangle ABC \cong \triangle XYZ \)

(2) \( YZ \parallel BC \), \( XZ \parallel AC \), \( XY \parallel AB \).

\[ \text{Proof: Outline.} \]

(i) \( \triangle ARO \cong \triangle BRO \)

(ii) \( \angle BOA \) is a parallelogram

(iii) \( \angle BOA \) is a rhombus

(iv) symmetric argument \( \Rightarrow \) \( \angle COA \) is a rhombus.

(v) \( BZ = CY \) \( \Rightarrow \) \( BZ \parallel CY \)

\( BZ \parallel CY \) \Rightarrow \( \{ \)

\( \Rightarrow \) \( BC \parallel YZ \)

\( BC = YZ \).

(vi) symmetric argument \( \Rightarrow \)

(a) \( AB \parallel XY \)

\( AB = XY \)

(b) \( AC \parallel XZ \)

\( AC = XZ \)

(vii) SSS \( \Rightarrow \) \( \triangle ABC \cong \triangle XYZ \).
(i) \( \triangle ARO \cong \triangle BRO \)
\[
\begin{align*}
AO & = BO \quad \text{HA} \\
RO & = RO
\end{align*}
\implies \triangle ARO \cong \triangle BRO.
\]

(ii) \( \square ZBOA \) is a parallelogram.
\( \triangle ARO \cong \triangle BRO \) \quad \text{corr. sides}
\implies AR = BR. \quad \text{But also, } OR = RZ.

It follows that quadrilateral \( \square ZBOA \) has diagonals \( OZ \) and \( AB \) that bisect each other; 1.9 \( \implies \square ZBOA \) is a parallelogram.

(iii) \( \square ZBOA \) is a rhombus.
Diagonals \( OZ \) and \( AB \) are perpendicular \( \implies \square ZBOA \) is a rhombus.

(iv) Asymmetrical argument \( \implies \square COAY \) is also a rhombus.

(v) \( \square BZCY \) is a parallelogram.
\( BZ = AO \) \quad \text{since rhombus } \square ZBOA \Rightarrow BZ = CY.
\( AO = CY \) \quad \text{COAY}

We also have:
\( BZ \parallel AO \) \quad \text{since rhombi} \Rightarrow BZ \parallel CY.
\( AO \parallel CY \)
Now, we have \( BZ = CY \) \( \Rightarrow \) \( BZ \parallel NC \) \( \Rightarrow \) \( BZYC \) is parallelogram.

\[ \begin{align*}
\frac{BC}{YZ} & \\frac{BC}{YZ} \\
\text{symmetric argument} & \\
\\
\text{(vii) SSS} & \Rightarrow \triangle ABC \cong \triangle XYZ.
\end{align*} \]

\[ \begin{align*}
\begin{cases}
AB = XY \\
AB \parallel XY
\end{cases} \\
\begin{cases}
AC = XZ \\
AC \parallel XZ
\end{cases}
\end{align*} \]