Math 575
Problem Set 9

1. Show that if $G$ is a weighted graph and $e$ is an edge whose weight is smaller than that of any other edge, then $e$ must belong to every minimum weight spanning tree for $G$.

Solution. Suppose that $T$ is a minimum weight spanning tree for $G$ that does not contain the edge $e$.
Then Consider the graph $T + e$.
This graph must contain a cycle $C$ that contains the edge $e$.
Let $f$ be an edge of $C$ different from $e$, and set $T^* = T + e - f$.

Then $T^*$ is also a spanning tree for $G$, but
$$\text{wt}(T^*) = \text{wt}(T + e - f) = \text{wt}(T) + \text{wt}(e) - \text{wt}(f) < \text{wt}(T),$$
contrary to $T$ being a minimum weight spanning tree.
Hence no such tree $T$ (i.e., without $e$) can exist.

2. Show that if all the weights of the weighted graph $G$ are distinct, then there is a unique minimum weight spanning tree for $G$.

Solution. The proof somewhat mimics that of the proof of Kruskal’s Algorithm.
Suppose that $T$ is a tree generated by Kruskal’s Algorithm (in fact, a moment’s thought shows that with the conditions of the problem, only one such tree could be generated). We claim there are no other minimum weight spanning trees for $G$. Suppose (and we will show this leads to a contradiction) that there are other minimum weight spanning trees, and choose one, $T'$.

Then suppose that $e$ is the first edge of $T$ that is not in $T'$. In other words, suppose that the edges of $T$, in the order they were added to form $T$, are $e_1, e_2, \ldots, e_k, \ldots, e_{n-1}$ and that $e = e_k$ and for all $i < k$, $e_i \in T'$. Let $C$ be the cycle in $T' + e$ that contains $e$. Let $f$ be an edge of $C$ that is not in $T'$. We note that by the nature of Kruskal’s algorithm, the weight of $f$ must be greater than the weight of $e$. This is because at the time we placed $e$ into $T$, $f$ was also available and would not have produced a cycle (since all the edges of $T$ up to that point are in $T'$ as well). So if $\text{wt}(f) < \text{wt}(e)$, we would have used $f$ at that juncture.

So now set $T^* = T' + e - f$ is a spanning tree of weight less than $T'$ - a contradiction.
3. Find two distinct minimum weight spanning trees for the graph below.

Solution Any spanning tree of weight 16 is a minimum weight spanning tree.

4. Find a minimum weight spanning tree for the graph below.

5. Prove: Every $k$-chromatic graph contains a copy of every tree on $k$ vertices.
   
   Proof. Since $G$ is a $k$-chromatic graph, $G$ contains a subgraph $H$ that is $k$-critical (just keep throwing away vertices until you can’t do it any longer). Then we know that $\delta(G) \geq k - 1$.

   Thus $H$ (and hence $G$ as well) contains every tree on $\delta(G) + 1 \geq k$ vertices.

6. Prove: If $G$ is a connected graph with no induced $P_4$, then $\chi(G) = \omega(G)$.

   Solution. By a previous exercise, we know that $G = A \oplus B$ for some two subgraphs $A$ and $B$ of $G$. The result now follows by induction and the fact that $\chi(G) = \chi(A) + \chi(B) = \omega(A) + \omega(B) = \omega(G)$. 