Math 575  
Problem Set 11

1. What is the connectivity of the complete bipartite graph $K_{r,s}$ (in terms of $r$ and $s$)?

   **Solution:** $\kappa(K_{r,s}) = \min\{r,s\}$.

2. Prove that if $v$ and $u$ are any two vertices in a 2-connected graph $G$, then there exist two internally disjoint $v$-$u$ paths in $G$.

   **Solution:** Since $v$ and $u$ must lie on a common cycle $C$, we get one path by following $v$ to $u$ along $C$ in a clockwise direction and then a second path by following $C$ from $v$ to $u$ in a counter-clockwise direction.

3. Prove that for any three vertices $v$, $u$ and $w$ in a 2-connected graph, there is
   (i). a $v$-$u$ path $P$ that does not contain $w$.
   (ii). there is also a $v$-$u$ path $Q$ that does contain $w$.

   **Solution:**
   (i). Since $G$ is 2-connected, $w$ cannot be a cut-vertex of $G$. Thus $G - w$ is connected. Let $P$ be any path in $G - w$. This path $P$ is a path in $G$ that does not contain $w$.

   (ii). Form a new graph $G^*$ by adding a new vertex $x$ that is adjacent to both $v$ and $u$.
   Then by the Asterisk Lemma, $G^*$ is also 2-connected. But then the vertices $x$ and $w$ must belong to a common cycle $C$ in $G^*$. But now $vx$ and $vu$ must be edges of $C$. Thus $C - x$ is a path in $G$ that contains $w$.

   **Lemma** (the Fan Lemma). If $G$ is $k$-connected, $A$ is a set of $k$ or more vertices in $G$, and $v$ is a vertex that is not in $A$, then there exists $k$ paths $P_1, P_2, ..., P_k$ with initial vertex $v$ and terminating respectively at some $a_j \in A$, and such that
   (i). No path $P_j$ contains a vertex of $A$ other than its terminal vertex $a_j$
   (ii). For all $i \neq j$, $V(P_i) \cap V(P_j) = \{v\}$.

4. Prove the Fan Lemma for the case $k = 2$.

   **Solution:** Let $G$ be a 2-connected graph and let $A$ be a set of two or more vertices of $G$. As in the previous problem, form a new graph $G^*$ by adding a new vertex $x$ adjacent to all the vertices of $A$.

   Then $G^*$ is 2-connected and there must be internally disjoint paths $P$ and $Q$ from $v$ to $x$ in $G^*$. Assume that we have chosen these so that they are as short as possible. Then each of $P$ and $Q$ must terminate in an edges $ax$, $bx$ where $a$ and $b$ are in $A$. And no other vertices of $P, Q$ can be in $A$ or else we could obtain shorter paths. This, $P' = P - x, Q' = Q - x$ are internally disjoint paths from $v$ to $A$ that satisfy the conditions of the Fan Lemma.
5. Show that if \( v, u \) and \( w \) are vertices in a 3-connected graph, then they lie on a common cycle.
[You may assume that every two vertices of a 2-connected graph lie on a common cycle, but you may not use Menger’s Theorem. However, you may use the Fan Lemma for \( k = 3 \).]

**Hint:** Since \( G \) is 2-connected, you may assume that there is a cycle \( C \) that contains \( v \) and \( u \). Now suppose that \( w \) does not lie on \( C \).