Solving Nonhomogeneous Linear DE's

**Theorem:** Suppose $I$ is an open interval, $p_1(x), \ldots, p_m(x)$, and $f(x)$ are continuous on $I$, $a \in I$, and $b_0, \ldots, b_{m-1} \in \mathbb{R}$. Consider the nonhomogeneous linear differential equation

$$(NH) \ y^{(m)} + p_1(x)y^{(m-1)} + \ldots + p_m(x)y = f(x)$$

and its associated

$$(AH) \ y^{(m)} + p_1(x)y^{(m-1)} + \ldots + p_m(x)y = 0.$$ 

Then the following hold:

(a) There exist $m$ linearly independent solutions $y_1, y_2, \ldots, y_m$ to $(AH)$ on $I$, and for any such $m$ linearly independent solutions $y_1, y_2, \ldots, y_m$ to $(AH)$ on $I$, the function

$$y_e(x) = c_1 y_1(x) + \ldots + c_m y_m(x),$$

where $c_1, \ldots, c_m$ are constants, is the general solution to $(AH)$ on $I$ (called the complementary function for $(NH)$).

(b) There exists a particular solution $y_p(x)$ to $(NH)$ on $I$, and for any solution $y_p(x)$ to $(NH)$ on $I$, the general solution to $(NH)$ on $I$ is $y(x) = y_e(x) + y_p(x)$.

(c) There exists exactly one solution $y(x)$ on $I$ to the initial value problem consisting of $(NH)$ and the initial conditions $y(a) = b_0, y'(a) = b_1, \ldots, y^{(m-1)}(a) = b_{m-1}$ (and $y(x)$ has the form $y_e(x) + y_p(x)$, for some choice of constants $c_1, \ldots, c_m$).

In our last class meeting we considered the homogeneous linear differential equation with real constant coefficients

$$(*) \ a_m y^{(m)} + a_{m-1} y^{(m-1)} + \ldots + a_1 y' + a_0 y = 0,$$

where $a_m \neq 0$, and its characteristic equation

$$(** \ a_m \lambda^n + a_{m-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$

and studied
Some ways to convert (* *) into a certain factored form (*) and then to use (*) to produce in linearly independent solutions $y_1, ..., y_m$ to (*) on $R$, and hence to obtain the general solution to (* *)

$y(x) = c_1 y_1(x) + ... + c_m y_m(x)$ on $R$, where $c_1, ..., c_m$ are constants.

We now study the Method of Undetermined Coefficients for finding a particular solution $y_p(x)$ to

$(NH) a_n y^{(m)} + a_{n-1} y^{(m-1)} + ... + a_1 y' + a_0 y = f(x)$, where $a_n \neq 0$, each $a_i$ is a real constant, and $f(x)$ is of a certain form.

Case I: Where no term of the trial solution appears in $y(x)$

Form of $f(x)$ Form of the trial solution $y_p(x)$

$P_m(x) = c_n x^m + ... + c_1 x + c_0$ $Q_m(x) = B_n x^m + ... + B_1 x + B_0$

$P_m(x) \in e^x$ $Q_m(x) \in e^x$

$P_m(x) \in e^{ax}$ $Q_m(x) \in e^{ax}$

$P_m(x) \in e^{ax} \cos(bx)$ $Q_m(x) \in e^{ax} \cos(bx) + (En^m \sin(-Fx)) e^{ax} \sin(bx)$

Case II: Where some term of the above trial solution appears in $y(x)$, and we denote it by $y_i(x)$.

Then for a trial solution $y_p(x)$, we use instead $y_p(x) = x y_i(x)$, where $i$ is the smallest positive integer so that $i$ has no term in common with $y_i$.

After obtaining the form of $y_p(x)$, one substitutes it into $(NH)$ and finds the actual values of the undetermined coefficients in $y_p(x)$ by equating the L.H.S. of $(NH)$ with $f(x)$. 