Initially a tank holds $V_0$ gal. of solution that contains $a$ lb. of a certain substance. Another solution containing $b$ lb. of the substance per gal. is poured into the tank at the rate of $e$ gal./min. while, simultaneously, the well stirred solution leaves the tank at the rate of $f$ gal./min.

**Problem:** Find the number of lb. of the substance in the tank at any time $t$.

**Solution:** Let $S$ be the substance and let $Q = Q(t) = \text{the amount (in lb.) of } S \text{ in the tank at any time } t \text{ (in min.)}, \ t \geq 0$. Then, since no amount of $S$ is created or destroyed in the process, the rate at which $Q$ changes in time is governed by

\[
\frac{dQ}{dt} = \text{(rate } S \text{ flows in)} - \text{(rate } S \text{ flows out}).
\]

Clearly, $S$ enters the tank at the rate $(b \text{ lb./gal.}) \times (e \text{ gal./min.)} = be \text{ lb./min.}$.

Now, at time $t$, the volume of the solution in the tank is $V_0 + et - ft$ gal., so the concentration of $S$ in the tank then is $\frac{Q(t)}{V_0 + (e - f)t}$ lb./gal. Thus the rate $S$ flows out = $\frac{Q(t)}{V_0 + (e - f)t} \times f$

lb./gal./x gal./min. = $\frac{fQ(t)}{V_0 + (e - f)t}$ lb./min.

To complete the solution, one solves the first order linear differential equation

\[
\frac{dQ}{dt} = be - \frac{fQ}{V_0 + (e - f)t}
\]

and initial condition $Q(0) = a$. 

1
Example 1: A tank initially holds 100 gal of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 gal/min, while the well stirred solution leaves the tank at the same rate. Find the amount of salt in the tank at any time $t$. (answer: $Q = 20e^{-t/20}$)

Example 2: A tank initially holds 100 gal of a brine solution containing 1 lb of salt. At $t = 0$ another brine solution containing 1 lb of salt per gal is poured into the tank at the rate of 3 gal/min, while the well stirred mixture leave the tank at the same rate. Find the amount of salt in the tank at any time $t$, and then find the time at which the mixture in the tank contains 2 lb of salt. (answer: $Q = 100 - 99e^{-0.03t}; t = -\frac{1}{0.03} \ln(98/99) \approx 0.388$ min)

Example 3: A tank contains 100 gal brine in which 10 lb salt is dissolved. Brine containing 2 lb salt/gal flows into the tank at 5 gal/min. If the well stirred mixture is drawn off at 4 gal/min, find the amount of salt in the tank at time $t$. (answer: $Q = 2(100 + t) - 190(100)^4(100 + t)^{-4}$)

Homework problems:

1. A 200-gal tank is full of water with 100 lb salt in solution. Pure water enters at 5 gal/min and the well stirred mixture runs out at the same rate. How much salt is present after 40 min? (answer: $Q = 100e^{-t/40}; 100/e \approx 36.8$ lb)

2. A tank contains 40 gal brine for which the concentration is initially 3 lb salt/gal. A salt solution of 2 lb salt per gal enters the tank at the rate of 5 gal/min and the well stirred mixture is drawn off at the same rate. Find the time required for the amount of salt in the tank to be reduced to 100 lb. (answer: $Q = 40e^{-t/8} + 80; t = 8 \ln 2 \approx 5.6$ min)

3. A tank contains 100 gal pure water. Brine containing 2 lb salt per gal is added to the tank at the rate of 5 gal/min. How much salt is present after 20 min if the well stirred mixture runs out at 6 gal/min? (answer: $Q = -\frac{200}{(100)^6}(100 - t)^6 + 2(100 - t); Q(20) = -200(\frac{4}{5})^6 + 160 \approx 107.6$ min)