Definition: If $A$ is a set and $x$ is an element of $A$, one writes “$x \in A$.”

Definition: By a complex number is meant an ordered pair $(a, b)$ of real numbers. Furthermore, the complex number $(a, b)$ is identified with, and sometimes referred to as, the point in the plane, $\mathbb{R} \times \mathbb{R}$, whose cartesian coordinates are $(a, b)$.

Definition: By the field $\mathbb{C}$ of complex numbers is meant the system $(\mathbb{R} \times \mathbb{R}, +, \cdot)$ where addition $+$ and multiplication $\cdot$ are defined by the following formulas. For all $(a, b), (c, d) \in \mathbb{C}$, $(a, b) + (c, d) = (a + c, b + d)$, and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

For simplicity, it is customary to modify the notation for a complex number $(a, b) \in \mathbb{R} \times \mathbb{R}$ and the binary operations $+$ and $\cdot$ in the following way: since $(a, b)$ can be uniquely represented in the form $a(1, 0) + b(0, 1)$, where $r(u, v)$ is defined to mean $(ru, rv)$, we reserve the letter “$i$” to denote the complex number $(0, 1)$, and we abbreviate the notation for $(1, 0)$ to simply the number “1” (since $(a, b) \cdot (1, 0) = (a, b) = (1, 0) \cdot (a, b)$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$). This permits the following notationally simplified definition of $\mathbb{C}$, which is the one we shall use from now on.

Definition: By the field $\mathbb{C}$ of complex numbers is meant the set of all expressions having the form $a + bi$, where $a, b \in \mathbb{R}$, endowed with the following operations of addition and multiplication (which are denoted with the same symbols used for addition and multiplication of real numbers): for all $a, b, c, d \in \mathbb{R}$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

With these operations on $\mathbb{C}$, $0 + 0i$, denoted 0, is its additive identity, and the additive inverse of $a + bi$ is $-a + (-b)i$, denoted $-a - bi$.

Its multiplicative identity is $1 + 0i$, denoted 1, and for $0 \neq a + bi \in \mathbb{C}$, the multiplicative inverse of $a + bi$, denoted $\frac{1}{a + bi}$, is easily shown to be $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} \cdot i$.

Notice that using the above formula for multiplication, one obtains $(0 + i) \cdot (0 + i) = (0 - 1) + (0 + 0)i$, and hence $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on. Some authors write the suggestive notation $i = \sqrt{-1}$, although the symbol “$\sqrt{-1}$” is considered meaningless or undefined when one is talking solely about $\mathbb{R}$.

Definition: For a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, $a$ is called the real part of $z$, $b$ is called the imaginary part of $z$, and the number $\sqrt{a^2 + b^2}$ is called the modulus of $z$ (or absolute value of $z$) and is denoted $|z|$.

Geometrically, we can think of the complex number $z = a + bi$, where $a, b \in \mathbb{R}$, as being represented by the point $(a, b)$ in the $xy$-plane. Then the modulus of $z$ provides the distance between the point $z$ and the origin. In this context, the $x$-axis is called the real axis, the $y$-axis is the imaginary axis, and the $xy$-plane is the complex plane.
**Definition:** If \( a, b \in \mathbb{R} \) and \( 0 \neq z = a + bi \) is a nonzero complex number, then we can also write \( z = |z| \left( \frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}} \cdot i \right) \), and since the point \( \left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right) \) is obviously on the unit circle in the plane, then there exists a real number \( \theta \) such that \( \cos \theta = \frac{a}{\sqrt{a^2 + b^2}} \) and \( \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \). Thus we may write \( z = r(\cos \theta + i \sin \theta) \), where \( r = |z| \). The latter is called the polar form of \( z \), and \( \theta \) is called an argument of \( z \). Hence, the rectangular coordinates of the point \( z \) in the plane are \((a, b)\), and a set of polar coordinates for \( z \) are \((r, \theta)\). As is the case for polar coordinates of any point, note that the argument \( \theta \) of \( z \) is not uniquely determined, since in the preceding representation for \( z \), \( \theta \) may be replaced by any real number having the form \( \theta + 2n\pi \), where \( n \) is an integer. So, \( \theta \) denotes the signed radian measure of any angle whose initial side is the positive \( x \)-axis and whose terminal side is the line segment from 0 to \( z \).

We shall use the following straightforward consequence of the formula for multiplication in \( \mathbb{C} \) and the formulas \( \cos(A + B) = \cos A \cos B - \sin A \sin B \) and \( \sin(A + B) = \sin A \cos B + \cos A \sin B \).

**Lemma:** If \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \) are complex numbers, then \( z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))) \).

Thus, to multiply two complex numbers together, we multiply their absolute values and add their angles. To add two complex numbers \( z_1, z_2 \), we view each complex number as being the tip of a position vector emanating from the origin, and then, geometrically, \( z_1 + z_2 \) is the tip of the vector sum \( z_1 + z_2 \).

In particular, one obtains the next result.

**DeMoivre’s Theorem:** If \( z = r(\cos \theta + i \sin \theta) \) is a complex number and \( n \) is a positive integer, then \( z^n = r^n(\cos(n\theta) + i \sin(n\theta)) \).

We shall use a corollary of DeMoivre’s Theorem to find roots of nonzero complex numbers.

**Definition:** If \( n \) is a positive integer and \( z, t \) are complex numbers such that \( t^n = z \), then \( t \) is said to be an \( n^{th} \) root of \( z \).

**Corollary:** If \( z = r(\cos \theta + i \sin \theta) \), where \( z \neq 0 \) and \( n \) is a positive integer, then for each nonnegative integer \( k \), \( z_k = r^{1/n} \left( \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right) \) is an \( n^{th} \) root of \( z \). Furthermore, \( z \) has exactly \( n \) distinct roots in \( \mathbb{C} \), namely, \( \{z_0, z_1, z_2, \ldots, z_{n-1}\} \).

**Exercise:** Let \( n \) be a positive integer. Let \( k \) be a nonnegative integer and define \( z_k \) by \( z_k = \cos \left( \frac{2\pi k}{n} \right) + i \sin \left( \frac{2\pi k}{n} \right) \). Show that \( z_k \) is an \( n^{th} \) root of 1.

Examples we have already seen: \(-1 \) and \( 1 \) are the square roots of 1, \(-i \) and \( i \) are the square roots of \(-1 \), and \(-1, i, -1 \) and \( 1 \) are the fourth roots of 1.

**Exercise:** Use the Corollary and the two equations \( 1 = 1 \cdot (\cos(0) + i \sin(0)) \) and \( -1 = 1 \cdot (\cos(\pi) + i \sin(\pi)) \) to generate the above square roots of \(-1 \) and 1, and the fourth roots of 1.
Exercise: Find formulas for all 3rd roots of 1 and all 6th roots of 1. Hint: use the above appropriate definitions of $z_k$, as well as trigonometry formulas such as $\cos(\pi/3) = 1/2 = \sin(\pi/6)$ and $\sin(\pi/3) = \sqrt{3}/2 = \cos(\pi/6)$, and similar formulas for $\cos \theta$ and $\sin \theta$, where $\theta$ denotes various integer multiples of $\pi/6$.

Exercise: Find formulas for all 4th roots of $-1$. Hint: use $-1 = \cos \pi = 1 \cdot (\cos \pi + i \sin \pi)$ and the above appropriate definitions of $z_k$, as well as trigonometry formulas such as $\cos(\pi/4) = \sqrt{2}/2 = \sin(\pi/4)$ and similar formulas for $\cos \theta$ and $\sin \theta$, where $\theta$ denotes various integer multiples of $\pi/4$.

Exercise: Find formulas for all 4th roots of $-16$. Hint: $-16 = 2^4(-1)$.

Using the above results about $C$ to help find solutions to homogeneous linear differential equations with constant coefficients:

Definition: The complex valued exponential function, denoted $e^z$, is defined in a way that its value at any real number agrees with the value given in courses on calculus and real numbers, and so that it has nice algebraic properties such as $e^{r+s} = e^r \cdot e^s$. As a result, one can derive, or take as a definition, the Euler formula, $e^{i\theta} = \cos \theta + i \sin \theta$, so that every nonzero complex number $z = r(\cos \theta + i \sin \theta)$ can be expressed as $z = re^{i\theta}$ or as $z = e^{\ln r + i\theta}$, where $r > 0$ and $\theta$ are real numbers.

Previously, we have shown that for a real number $r$, $e^{rx}$ is a solution to a second order homogeneous differential equation with real constant coefficients ($\star$) $ay'' + by' + cy = 0$ iff $r$ is a root to the characteristic equation ($\star\star$) $ar^2 + br + c = 0$ of ($\star$), and we have shown that if $b^2 - 4ac > 0$, then two linearly independent solutions to ($\star$) are $e^{r_1x}$ and $e^{r_2x}$, where $r_1$ and $r_2$ are the two distinct (real) roots to ($\star\star$). We also learned that if $b^2 - 4ac < 0$ (referred to as Case III), then two linearly independent solutions to ($\star$) are the functions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$, where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. Using the previous definition, one can obtain the following.

Theorem: Let ($\star$) and ($\star\star$) be as above, where $b^2 - 4ac < 0$. Then, in $\mathbb{C}$, for any real numbers $\alpha$ and $\beta$, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are roots of ($\star\star$) if and only if $e^{r_1x} = e^{(\alpha + i\beta)x}$ and $e^{r_2x} = e^{(\alpha - i\beta)x}$ are solutions to ($\star$).

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, and hence $e^{(\alpha \pm i\beta)x}$ are the functions $e^{\alpha x}(\cos \beta x + i \sin \beta x)$ and $e^{\alpha x}(\cos \beta x - i \sin \beta x)$, then the above theorem also leads us to the (real) linearly independent solutions to ($\star$) that we obtained in our class, namely, $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$, where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. One can see this by noting that linear combinations of the functions in the preceding theorem produce the real valued solutions given in class, since $1/2(e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x})$ produces $e^{\alpha x} \cos \beta x$, and $1/2i$ times $e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}$ produces $e^{\alpha x} \sin \beta x$.

Exercise: Find four linearly independent, real valued solutions to the homogeneous linear differential equation ($\star$) $y^{(4)} + 16y = 0$. Hint: The characteristic equation of ($\star$) is ($\star\star$) $r^4 + 16 = 0$. For two of the 4th roots $\alpha + i\beta$ and $-\alpha + i\beta$ of ($\star\star$), i.e., of the number $-16$, form the functions $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$, $e^{-\alpha x} \cos \beta x$, and $e^{-\alpha x} \sin \beta x$. (There is no need to use the other roots of ($\star\star$) since $\cos(-\beta x) = \cos \beta x$ and $\sin(-\beta x) = -\sin \beta x$.)