Topic Course on Probabilistic Methods
(Week 5)
Lovász Local Lemma

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Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
The second moment method

- Lovász Local Lemma
- Property B
- \( k \)-coloring of \( \mathbb{R} \)
- Ramsey numbers \( R(k, k) \)
- Ramsey numbers \( R(3, k) \)
- Directed cycles
- Linear Arboricity
Lovász Local Lemma

- $A_1, A_2, \ldots, A_n$: $n$ events in an arbitrary probability spaces.
Lovász Local Lemma

- \( A_1, A_2, \ldots, A_n \): \( n \) events in an arbitrary probability spaces.
- A dependency digraph \( D = (V, E) \): if for each \( A_i, A_i \) is mutually independent to all the events \( \{A_j : A_i A_j \notin E\} \).

**Lovász Local Lemma, general case:** If there are real number \( x_1, \ldots, x_n \) such that \( 0 \leq x_i < 1 \) and 
\[
\Pr(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j)
\]
for all \( 1 \leq i \leq n \). Then
\[
\Pr \left( \bigwedge_{i=1}^{n} \bar{A}_i \right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.
\]
Proof: Inductively prove that for any $S \subset [n]$, $|S| = s < n$, $i \notin S$,

$$\Pr \left[ A_i \mid \bigwedge_{j \in S} \bar{A}_j \right] \leq x_i.$$
Proof: Inductively prove that for any $S \subset [n]$, $|S| = s < n$, $i \not\in S$,

$$\Pr \left[ A_i \mid \bigwedge_{j \in S} \overline{A}_j \right] \leq x_i.$$  

Trivial for $s = 0$. Assuming it for all $s' < s$, we prove it for $s$. 
Proof: Inductively prove that for any \( S \subset [n], |S| = s < n,\) \( i \notin S,\)

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\Pr \left[ A_i \mid \land_{j \in S} \bar{A}_j \right] \leq x_i.
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Let \( S_1 = \{j \in S: (i, j) \in E(G)\} \) and \( S_2 = S \setminus S_1.\) Then
Proof: Inductively prove that for any $S \subset [n]$, $|S| = s < n$, $i \notin S$,

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Let $S_1 = \{j \in S: (i, j) \in E(G)\}$ and $S_2 = S \setminus S_1$. Then

$$\Pr \left[ A_i \mid \bigwedge_{j \in S} \bar{A_j} \right] = \frac{\Pr \left[ A_i \land \left( \bigwedge_{j \in S} \bar{A_j} \right) \mid \bigwedge_{j \in S} \bar{A_j} \right]}{\Pr \left[ \bigwedge_{j \in S} \bar{A_j} \mid \bigwedge_{j \in S} \bar{A_j} \right]}$$
Proof: Inductively prove that for any $S \subset [n], \ |S| = s < n$, $i \not\in S$,

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$$\Pr \left[ A_i \mid \bigwedge_{j\in S} \overline{A}_j \right] = \frac{\Pr \left[ A_i \wedge \left( \bigwedge_{j\in S} \overline{A}_j \right) \mid \bigwedge_{j\in S} \overline{A}_j \right]}{\Pr \left[ \bigwedge_{j\in S} \overline{A}_j \mid \bigwedge_{j\in S} \overline{A}_j \right]}$$

$$\Pr \left[ A_i \wedge \left( \bigwedge_{j\in S} \overline{A}_j \right) \mid \bigwedge_{j\in S} \overline{A}_j \right] \leq \Pr \left[ A_i \mid \bigwedge_{j\in S} \overline{A}_j \right]$$

$$= \Pr[A_i] \leq x_i \prod_{(i,j) \in E(G)} (1 - x_j).$$
Write \( S_1 = \{j_1, j_2, \ldots, j_r\} \).

\[
\Pr \left[ \bigwedge_{j \in S} \bar{A}_j \mid \bigwedge_{j \in S} \bar{A}_j \right] \\
= \prod_{l=1}^{r} \left( 1 - \Pr \left[ A_{j_l} \mid \bar{A}_{j_{l+1}} \land \cdots \land A_{j_r} \land j \in S \bar{A}_j \right] \right) \\
\geq \prod_{l=1}^{r} (1 - x_{j_l}) \\
\geq \prod_{(i,j) \in E(G)} (1 - x_j).
\]

Thus,
\[
\Pr \left[ A_i \mid \bigwedge_{j \in S} \bar{A}_j \right] \leq x_i.
\]
\[
\Pr \left[ \bigwedge_{i=1}^{n} \bar{A}_i \right] = (1 - \Pr[A_1])(1 - \Pr[A_2|\bar{A}_1]) \cdots \\
\cdots (1 - \Pr[A_n|\bigwedge_{i=1}^{n-1} \bar{A}_i]) \\
\geq \prod_{i=1}^{n} (1 - x_i).
\]

The proof is finished.}\]

\[\square\]
Lovász Local Lemma, symmetric case: Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent of a set of all the other event $A_j$ but at most $d$, and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d + 1) < 1$, then $\Pr(\bigwedge_{i=1}^{n} \overline{A_i}) > 0$. 

Symmetric Case
Theorem: Let $H = (V, E)$ be a hypergraph in which every edge has at least $k$ elements, and suppose that each edge of $H$ intersects at most $d$ other edges. If $e(d + 1) \leq 2^{k-1}$, then $H$ has property $B$. 
Theorem: Let $H = (V, E)$ be a hypergraph in which every edge has at least $k$ elements, and suppose that each edge of $H$ intersects at most $d$ other edges. If $e(d + 1) \leq 2^{k-1}$, then $H$ has property $B$.

Proof: Color each vertex in two colors randomly and independently. For each edge $f \in E$, let $A_f$ be the event that $f$ is monochromatic. Then

$$\Pr(A_f) = 2^{1-|f|} \leq 2^{1-k}.$$

$A_f$ is independent to all event but at most $d$. Aply LLL. □
Let \( c : \mathbb{R} \to \{1, 2, \ldots, k\} \) be a \( k \)-coloring of \( \mathbb{R} \). A set \( T \subset \mathbb{R} \) is **multicolored** if \( c(T) = \{1, 2, \ldots, k\} \).
Let $c: \mathbb{R} \rightarrow \{1, 2, \ldots, k\}$ be a $k$-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \ldots, k\}$.

**Theorem:** Let $m$ and $k$ be two positive integers satisfying
\[
e^m e^k \left( m - 1 \right) + 1 \leq 1.
\]

Then, for any set $S$ of $m$ real numbers there is a $k$-coloring so that each translation $x + S$ (for $x \in \mathbb{R}$) is multicolored.
**$k$-coloring of $\mathbb{R}$**

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**Theorem:** Let $m$ and $k$ be two positive integers satisfying

$$e(m(m - 1) + 1)k(1 - \frac{1}{k})^m \leq 1.$$

Then, for any set $S$ of $m$ real numbers there is a $k$-coloring so that each translation $x + S$ (for $x \in \mathbb{R}$) is multicolored. The condition is satisfied if $m \geq (3 + o(1))k \log k$. 
First we use LLL to prove “For any finite set $X \subset \mathbb{R}$, there is a $k$-coloring so that $x + S$ (for all $x \in X$) is multi-colored.”
Proof

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Let $Y = \bigcup_{x \in X} (x + S)$. Color numbers in $Y$ in $k$-colors randomly and independently. Let $A_x$ be the event that $x + S$ is not multi-colored.

$$\Pr(A_x) \leq k(1 - \frac{1}{k})^{m-1}.$$
Proof

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$$\Pr(A_x) \leq k \left(1 - \frac{1}{k}\right)^{m-1}.$$ 

$A_x$ depends on $A_y$ if $(x + S) \cap (y + S) \neq \emptyset$. Equivalently, $y - x \in S - S$. There are at most $m(m - 1)$ such events.

$$d \leq m(m - 1).$$
Applying LLL, we get

\[ \Pr(\land_{x \in X} \bar{A}_x) > 0. \]

Then by Tikhonov’s theorem, \([k]^{\mathbb{R}}\) is compact. For any \(x \in \mathbb{R}\), let

\[ C_x = \{ c \in [k]^{\mathbb{R}} : x + S \text{ is multi-colored} \}. \]
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\[ C_x = \{ c \in [k]^{\mathbb{R}} : x + S \text{ is multi-colored} \}. \]

Now \(C_x\) is a closed set and \(\bigcap_{x \in X} C_x \neq \emptyset\) for any finite \(X\). Then \(\bigcap_{x \in \mathbb{R}} C_x \neq \emptyset\). \(\square\)
Theorem (Spencer, 1975)

\[ R(k, k) \geq (1 + o(1)) \frac{\sqrt{2}}{e} k^{2^{k/2}}. \]
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Ramsey numbers

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\[ R(k, k) \geq (1 + o(1)) \frac{\sqrt{2}}{e} k^{2k/2}. \]

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Best bounds for \( R(r, k) \) (for fixed \( r \) and \( k \) large),

\[ c \left( \frac{k}{\log k} \right)^{(r+1)/2} < R(r, k) < (1 + o(1)) \frac{k^{r-1}}{\log^{r-2} k}. \]
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Proof: Consider \( G(n, p) \). Two bad events:

- For \( S \in \binom{[n]}{3} \), let \( A_S \) be the event of \( G|_S \) is a triangle;
  \[ \Pr(A_S) = p^3. \]
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- For \(T \in \binom{[n]}{k}\), let \(B_T\) be the event that \(T\) is an independent set of \(G\); \(\Pr(B_t) = (1 - p)^{\binom{k}{2}}\).
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- For \( T \in \binom{[n]}{k} \), let \( B_T \) be the event that \( T \) is an independent set of \( G \); \( \Pr(B_t) = (1 - p)^{\binom{k}{2}} \).
- Dependence graph: \( d_{SS} \leq 3n \), \( d_{ST} \leq 3\binom{n}{k-2} \), \( d_{TS} \leq \binom{k}{2}n \), and \( d_{TT} \leq \binom{k}{2}\binom{n}{k-2} \).
Proof

By LLL, we only require

\[ p^3 \leq x(1 - x)^{3n}(1 - y)^3 \binom{n}{k-2} \]

\[ (1 - p)^k \leq y(1 - x)^n(1 - y)^k \binom{k}{k-2}. \]
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\[ (1 - p)^\binom{k}{2} \leq y(1 - x)^\binom{k}{2}n(1 - y)^\binom{k}{2}\binom{n}{k-2}. \]

We can choose \( p = c_1n^{-1/2}, k = c_2n^{1/2}\log n, x = c_3n^{-3/2}, \)
and \( y = c_4/\binom{n}{k}. \)
Proof

By LLL, we only require

\[ p^3 \leq x(1-x)^{3n}(1-y)^{3\binom{n}{k-2}} \]

\[ (1-p)^{\binom{k}{2}} \leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{k}{2}\binom{n}{k-2}}. \]

We can choose \( p = c_1 n^{-1/2}, \ k = c_2 n^{1/2} \log n, \ x = c_3 n^{-3/2} \),
and \( y = c_4 / \binom{n}{k} \).

This gives \( R(3, k) > c_5 k^2 / \log^2 k \). \qed
Best bounds for $R(r, k)$ (for fixed $r$ and $k$ large),

$$c \left( \frac{k}{\log k} \right)^{(r+1)/2} < R(r, k) < (1 + o(1)) \frac{k^{r-1}}{\log^{r-2} k}.$$

**Erdős conjecture $250$:** Prove

$$R(4, k) > c' \frac{k^3}{\log^c k}$$

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The best lower bound is using LLL; $R(4, k) > c' \frac{k^{2.5}}{\log^{2.5} k}$.
Directed cycles

- $D = (V, E)$: a simple directed graph.
- $\delta$: minimum outdegree.
- $\Delta$: maximum indegree.
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**Theorem [Alon and Linial (1989)]** If 
$$e(\Delta \delta + 1)(1 - 1/k)^\delta < 1,$$
then $D$ contains a (directed, simple) cycle of length $0 \mod k$. 
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**Theorem [Alon and Linial (1989)]** If $\varepsilon(\Delta \delta + 1)(1 - 1/k)^\delta < 1$, then $D$ contains a (directed, simple) cycle of length $0 \mod k$.

**Proof:** First we can assume every out-degree is $\delta$ by deleting some edges if necessary. Consider $f : V \rightarrow \mathbb{Z}_k$. Bad event $A_v$: no $u \in \Gamma^+(v)$ with $f(u) = f(v) + 1$.

$$\Pr(A_v) = (1 - 1/k)^\delta.$$  
Each event depends on at most $\delta \Delta$ others. Apply LLL. □
Linear Arboricity

- Linear forest: disjoint union of paths.
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The Linear Arboicity Conjecture (Akiyama, Exoo, Harary [1981]): For every $d$-regular graph $G$,

$$la(G) = \left\lceil \frac{d + 1}{2} \right\rceil.$$
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If the conjecture is true, then it is tight.

$$\text{la}(G) \geq \frac{nd}{2(n - 1)} > \frac{d}{2}.$$
Directed graphs

- $G = (V, E)$: a directed graph.
- $G$ is $d$-regular if $d^+(v) = d^-(v) = d$ for any vertex $v$.
- Linear directed forest: disjoint union of directed paths.
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The Linear Arboricity Conjecture for directed graph (Nakayama, Peroche [1981]): For every $d$-regular directed graph $G$, $dla(G) = d + 1$. 
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**The Linear Arboricity Conjecture for directed graph (Nakayama, Peroche [1981]):** For every \( d \)-regular directed graph \( G \), \( \text{dla}(G) = d + 1 \).

DLA conjecture for \( d \) implies LA conjecture for \( 2d \).
Proposition: Let $H = (V, E)$ be a graph with maximum degree $d$, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of $V$. If $|V_i| \geq 2ed$, then there is an independent set of vertices $W$ that contains a vertex from each $V_i$. 

A proposition
Proposition: Let $H = (V, E)$ be a graph with maximum degree $d$, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of $V$. If $|V_i| \geq 2ed$, then there is an independent set of vertices $W$ that contains a vertex from each $V_i$.

Proof: WLOG, we assume

$$|V_1| = |V_2| = \cdots = |V_r| = \lceil 2ed \rceil = g.$$ 

Pick from each $V_i$ a vertex randomly and independently. Let $W$ be the random set of the vertices picked. For each edge $f$, let $A_f$ be the event that both ends in $W$. The maximum degree in the dependence graph is at most $2gd - 1$. We have $e \cdot 2gd \cdot \frac{1}{g^2} = \frac{2ed}{g} < 1$. Apply LLL. □
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**Theorem** Let $G = (U, F)$ be a $d$-regular digraph with directed girth $g \geq 8ed$. Then

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**Theorem** Let $G = (U, F)$ be a $d$-regular digraph with directed girth $g \geq 8ed$. Then

$$\text{dla}(G) = d + 1.$$  

**Proof:** Using Hall’s matching theorem, we can partition $F$ into $d$ pairwise disjoint 1-regular spanning subgraphs $F_1, \ldots, F_d$ of $G$. 
Each $F_i$ is a union of vertex disjoint directed cycles. Let $V_1, \ldots, V_r$ are the sets of edges of all cycles. Then

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Apply the proposition to the line-graph $H$ of $G$. Note $H$ is $4d - 2$-regular.
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Apply the proposition to the line-graph $H$ of $G$. Note $H$ is $4d - 2$-regular.

There exists an independent set $M_1$ of $H$. Now $M_1, F_1 \setminus M_1, \ldots, F_d \setminus M_1$ forms $d + 1$ linear directed forests. □
**Theorem [Alon 1988]** There is an absolute constant $c > 0$ such that for every $d$-regular directed graph $G$

$$\text{dla}(G) \leq d + cd^{3/4} \log^{1/2} d.$$
General $d$-regular graphs

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**Corollary** There is an absolute constant $c > 0$ such that for every $d$-regular graph $G$

$$\text{dla}(G) \leq \frac{d}{2} + cd^{3/4} \log^{1/2} d.$$
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The error terms can be improved to $cd^{2/3} \log^{1/3} d$. 


Proof

Pick a prime $p$. Color each vertex randomly and uniformly into $p$ colors. I.e., consider a random map

$$f : V \rightarrow \mathbb{Z}_p.$$
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Pick a prime $p$. Color each vertex randomly and uniformly into $p$ colors. I.e., consider a random map

$$ f : V \to \mathbb{Z}_p. $$

Define for $i \in \mathbb{Z}_p$,

$$ E_i = \{ (u, v) \in E : f(v) = f(u) + i \}. $$

Let $G_i = (V, E_i)$ and

- $\Delta_i^+:$ the maximum out-degree of $G_i$.
- $\Delta_i^-:$ the maximum in-degree of $G_i$.
- $\Delta_i:$ the maximum of $\Delta_i^+$ and $\Delta_i^-$. 
There exists a $f$ satisfying

- All $G_i$ are almost regular: $\Delta_i \leq \frac{d}{p} + 3\sqrt{d/p}\sqrt{\log d}$. 
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- All $G_i$ can be completed to a $\Delta_i$-regular directed graph without decreasing the girth.
There exists a $f$ satisfying

- All $G_i$ are almost regular: $\Delta_i \leq \frac{d}{p} + 3 \sqrt{\frac{d}{p} \log d}$.
- $G_i$ has large girth $\geq p$ for $i \neq 0$.
- All $G_i$ can be completed to a $\Delta_i$-regular directed graph without decreasing the girth.

$$\text{dla}(G) \leq 2\Delta_0 + \sum_{i=1}^{p-1} (\Delta_i + 1) \leq d + \frac{d}{p} + p + C \sqrt{dp \log d}.$$  

Now choose $p \sim d^{1/2}$. 