Topic Course on Probabilistic Methods
(Week 2)
Linearity of Expectation (2)

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Introduction

Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
Linearity of Expectation (2)

- Disjoint pairs
- \( k \)-sets
- Balancing vectors
- Unbalancing lights
- Brégman’s Theorem
- Hamilton paths
- Independence number
- Turán Theorem
**Disjoint pairs**

- $\mathcal{F} \subseteq 2^{[n]}$.
- $d(\mathcal{F}) := |\{(F, F') : F, F' \in \mathcal{F}, F \cap F' = \emptyset\}|$. 


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**Theorem [Alon-Frankl, 1985]:** If $|\mathcal{F}| = 2^{(1/2+\delta)n}$, then

$$d(\mathcal{F}) < |\mathcal{F}|^{2 - \delta^2/2}.$$
Proof

Let \( m := 2^{(1/2+\delta)n} \). Suppose \( d(\mathcal{F}) < m^{2-\delta^2/2} \).
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Pick independently $t$ members $A_1, A_2, \ldots, A_t$ of $\mathcal{F}$ with repetitions at random.
Proof

Let \( m := 2^{(1/2+\delta)n} \). Suppose \( d(\mathcal{F}) < m^{2-\delta^2/2} \).

Pick independently \( t \) members \( A_1, A_2, \ldots, A_t \) of \( \mathcal{F} \) with repetitions at random.

\[
\Pr(| \bigcup_{i=1}^{t} A_i | \leq \frac{n}{2}) \\
\leq \sum_{|S| = \frac{n}{2}} \Pr(\bigwedge_{i=1}^{t} (A_i \subset S)) \\
\leq 2^n \left( \frac{2^{n/2}}{2^{(1/2+\delta)n}} \right)^t \\
= 2^n(1-\delta t).
\]
Let $v(B) = |\{ A \in \mathcal{F} : B \cap A = \emptyset \}|$. Then

$$
\sum_B v(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.
$$
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Let $Y$ be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all $A_i$ $1 \leq i \leq t$. 

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Let \( Y \) be a random variable whose value is the number of members \( B \in \mathcal{F} \) that is disjoint to all \( A_i \) \( 1 \leq i \leq t \). Then

\[
E(|Y|) = \sum_{B \in \mathcal{F}} \left( \frac{v(B)}{m} \right)^t \\
\geq \frac{1}{m^{t-1}} \left( \frac{\sum_B v(B)}{m} \right)^t \\
\geq 2m^{1-t\delta^2/2}.
\]
Since $Y \leq m$, we get

$$\Pr(Y \geq m^{1-t\delta^2/2}) \geq m^{-t\delta^2/2}.$$
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Choose $t = \lceil 1 + \frac{1}{\delta} \rceil$. We have $m^{-t\delta^2/2} > 2^{n(1-\delta t)}$.

Thus, with positive probability, $\left| \bigcup_{i=1}^{t} A_i \right| > \frac{n}{2}$ and $\bigcup_{i=1}^{t} A_i$ is disjoint to more than $2^{n/2}$ members of $\mathcal{F}$. Contradiction. □
Let $X_1, X_2, \ldots, X_n$ be random variables and 

$$X = \sum_{i=1}^{n} c_i X_i.$$ 

Then 

$$\mathbb{E}(X) = \sum_{i=1}^{n} c_i \mathbb{E}(X_i).$$
Let $X_1, X_2, \ldots, X_n$ be random variables and $X = \sum_{i=1}^{n} c_i X_i$. Then

$$E(X) = \sum_{i=1}^{n} c_i E(X_i).$$

**Philosophy:** There is a point in the probability space for which $X \geq E(X)$ and a point for $X \leq E(X)$. 
**Theorem:** Let \( G = (V, E) \) be a graph with \( n \) vertices and \( m \) edges. Then \( G \) contains a bipartite subgraph with at least \( m/2 \) edges.
Theorem: Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Then $G$ contains a bipartite subgraph with at last $m/2$ edges.

Proof: Consider a random partition $L \cup R$ of $V$ as follows. For each vertex $v$, put $v$ into $L$ or $R$ with equal probability.
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Let $X$ be the number of crossing edges (from $L$ to $R$). Let $X_{uv}$ be the indicator variable of the edge $uv$ is crossing.

$$\mathbb{E}(X_{uv}) = \frac{1}{4}.$$
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\( k \)-sets

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$k$-sets

- $V = V_1 \cup V_2 \cup \cdots \cup V_k$: a partition of equal parts, where $|V_1| = \cdots = |V_k| = n$.
- $h: V^k \to \{-1, 1\}$. 
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- For \( S \subset V \), let \( h(S) = \sum_{F \subset S} h(F) \).
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- For $S \subset V$, let $h(S) = \sum_{F \subset S} h(F)$.
- A $k$-set $F$ is crossing if it contains precisely one point from each $V_i$. 
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A \( k \)-set \( F \) is crossing if it contains precisely one point form each \( V_i \).

**Theorem:** Suppose \( h(F) = +1 \) for all crossing \( k \)-sets \( F \). Then there is an \( S \subset V \) for which

\[ |h(S)| \geq c_k n^k. \]

Here \( c_k > 0 \), independent of \( n \).
**Lemma:** Let $P_k$ be the set of all homogeneous polynomials $f(p_1, \ldots, p_k)$ of degree $k$ with all coefficients having absolute value at most one and $p_1 p_2 \cdots p_k$ having coefficient one. Then for all $f \in P_k$ there exists $p_1, \ldots, p_k \in [0, 1]$ with

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Proof: Let $M(f) = \max_{p_1, \ldots, p_k} |f(p_1, \ldots, p_k)|$. Note $P_k$ is compact and $M$ is continuous. $M$ reaches its minimum value $c_k$ at some point $f_0$. We have

$$c_k = M(f_0) > 0. \quad \square$$
Proof of theorem

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$$X_F = \begin{cases} 
  h(F) & \text{if } F \subset S, \\
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Say $F$ has type $(a_1, \ldots, a_k)$ if $|F \cap V_i| = a_i$, $1 \leq i \leq k$. 
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Say $F$ has type $(a_1, \ldots, a_k)$ if $|F \cap V_i| = a_i$, $1 \leq i \leq k$. For these $F$,

$$\mathbb{E}(X_F) = h(F)p_1^{a_1} \cdots p_k^{a_k}.$$
\[
E(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_{1}^{a_{1}} \cdots p_{k}^{a_{k}} \sum_{F \text{ of type } (a_1, \ldots, a_k)} h(F).
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Let \( f(p_1, \ldots, p_k) = \frac{1}{n^k} E(X) \). Then \( f \in P_k \).

Now select \( p_1, \ldots, p_k \in [0, 1] \) with \( |f(p_1, \ldots, p_k)| \geq c_k \).
Then \( E(|X|) \geq |E(X)| \geq c_k n^k \).
\[ \mathbb{E}(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_1^{a_1} \cdots p_k^{a_k} \sum_{F \text{ of type } (a_1, \ldots, a_k)} h(F). \]

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Then \( \mathbb{E}(|X|) \geq |\mathbb{E}(X)| \geq c_k n^k \).

There exists a \( S \) such that \( |h(S)| \geq c_k n^k \). \( \square \)
Balancing vectors

**Theorem:** Let $v_1, \ldots, v_n$ are $n$ unit vector in $\mathbb{R}^n$. Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n},$$

and also there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \geq \sqrt{n}.$$
Proof

Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from \{+1, −1\}. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$. 
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$$
E(X) = E\left(\sum_{i,j=1}^{n} \epsilon_i \epsilon_j v_i \cdot v_j \right)
$$

$$
= \sum_{i,j=1}^{n} E(\epsilon_i \epsilon_j) v_i \cdot v_j
$$

$$
= \sum_{i,j=1}^{n} \delta_{ij} v_i \cdot v_j
$$

$$
= \sum_{i=1}^{n} \|v_i\|^2 = n.
$$
Theorem: Let $v_1, \ldots, v_n \in \mathbb{R}^n$, all $\|v_i\| \leq 1$. Let $p_1, p_2, \ldots, p_n \in [0, 1]$ be arbitrary and set $w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ so that setting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$,

$$\|w - v\| \leq \frac{\sqrt{n}}{2}.$$
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Hint: Pick $\epsilon_i$ independently with

$$\Pr(\epsilon_i = 1) = p_i, \quad \Pr(\epsilon_i = 0) = 1 - p_i.$$ 

The proof is similar.
Theorem: Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exist $x_i, y_j = \pm 1, 1 \leq i, j \leq n$ so that

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left( \sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.$$
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**Proof:** Choose $y_j = 1$ or $-1$ randomly and independently. Let $R_i = \sum_{i=1}^{n} a_{ij} y_j$. Let $x_i$ be the sign of $R_i$. Then

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} |R_i|.$$
Each $R_i$ has the distribution $S_n = \sum_{i=1}^{n} X_i$, where $X_i$'s are independent uniform $\{-1, 1\}$ random variables.
Each $R_i$ has the distribution $S_n = \sum_{i=1}^{n} X_i$, where $X_i$’s are independent uniform $\{-1, 1\}$ random variables. We have

$$E(|S_n|) = n 2^{1-n} \left( \frac{n-1}{n-1} \right) \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$= \left( \sqrt{\frac{2}{\pi}} + o(1) \right) n^{1/2}.$$
Each $R_i$ has the distribution $S_n = \sum_{i=1}^{n} X_i$, where $X_i$’s are independent uniform $\{-1, 1\}$ random variables. We have

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$$= \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{1/2}.$$ 

Hence,

$$\sum_{i=1}^{n} E(|R_i|) = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}.$$
Brégman’s Theorem

- $A = (a_{i,j})$: an $n \times n$ matrix with all $a_{i,j} \in \{0, 1\}$. 
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- \( A = (a_{ij}) \): an \( n \times n \) matrix with all \( a_{i,j} \in \{0, 1\} \).
- \( S \): the set of permutations \( \sigma \in S_n \), with \( a_{i,\sigma(i)} = 1 \) for all \( i \).
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- \( \text{per}(A) = |S| \): the permanent of \( A \).
- \( r_i \): the \( i \)-th row sum.

Brégman’s Theorem (1973): \( \text{per}(A) \leq \prod_{1 \leq i \leq n} (r_i!)^{1/r_i} \).
Proof [Schrijver 1978]

Pick $\sigma \in S$ and $\tau \in S_n$ independently and uniformly.
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- Let $A^{(1)} := A$; and $A^{(i)}$ is the submatrix obtained by deleting row $\tau(i - 1)$ and column $\sigma(\tau(i - 1))$ for $2 \leq i \leq n$. 
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- $R_{\tau(i)}$: the $\tau(i)$’s row sum of $A^{(i)}$. 
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- $L = L(\sigma, \tau) := \prod_{i=1}^{n} R_{\tau(i)}$. 
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- $G(L) := e^{E(\ln L)} = e^{\sum_{i=1}^{n} E(\ln R_{\tau(i)})}$. 

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**Claim:** $\text{per}(A)) \leq G(L)$. 
For any fixed $\tau$. Assume $\tau(1) = 1$. By re-ordering, assume
the first row has ones in the first $r := r_1$ columns. For
$1 \leq j \leq r$ let $t_j$ be the permanent of $A$ with the first row
and $j$-th column removed (i.e., $\sigma(1) = j$). Let
\[
t = \frac{t_1 + \cdots + t_r}{r} = \frac{\text{per}(A)}{r}.
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$$t = \frac{t_1 + \cdots + t_r}{r} = \frac{\text{per}(A)}{r}.$$ 

By induction,

$$G(R_2 \cdots R_n | \sigma(1) = j) \geq t_j.$$ 

$$G(L) \geq \prod_{j=1}^{r} (rt_j)^{t_j/\text{per}(A)} = r \prod_{j=1}^{r} (t_j)^{t_j/r}.$$
Since \( \left( \prod_{j=1}^{r} t_{j}^{t_{j}} \right)^{\frac{1}{r}} \geq t^{t} \), we have

\[
G(L) \geq r \prod_{j=1}^{r} t_{j}^{t_{j}/rt} \geq r(t^{t})^{1/t} = rt = \text{per}(A).
\]
Since \( \left( \prod_{j=1}^{r} t_{j}^{t \cdot j} \right)^{1/r} \geq t^t \), we have

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Now we calculate \( G[L] \) conditional on a fixed \( \sigma \). By reordering, assume \( \sigma(i) = i \) for all \( i \). Note

\[
G(R_i) = (r_i!)^{1/r_i}.
\]
Since \( \left( \prod_{j=1}^{r} t_{j}^{t_{j}} \right)^{1/r} \geq t^t \), we have

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\[
G(R) = G\left( \prod_{i=1}^{n} R_i \right) = \prod_{i=1}^{n} (r_i!)^{1/r_i}.
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**Proof:** Let $X$ be the number of Hamiltonian paths in a random tournament. Write $X = \sum_{\sigma \in S_n} X_{\sigma}$. Here $X_{\sigma}$ is the indicator random variable for $\sigma$ giving a Hamilton path.

$$E(X_{\sigma}) = 2^{-(n-1)}.$$
**Theorem:** There is a tournament $T$ with $n$ players and at least $n!2^{-(n-1)}$ Hamiltonian paths.

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We have

$$E(X) = \sum_{\sigma \in S_n} E(X_\sigma) = n!2^{1-n}.$$ 

Done! \qed
Let $P(n)$ be the maximum possible number of Hamiltonian paths in a tournament on $n$ vertices.
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**Szele [1943] proved**

\[
\frac{1}{2} \leq \lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} \leq \frac{1}{2^{3/4}}.
\]

He conjecture that $\lim_{n \to \infty} \left( \frac{P(n)}{n!} \right)^{1/n} = \frac{1}{2}$. 
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**Szele [1943] proved**

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**Theorem [Alon, 1990]:** $P(n) \leq cn^{3/2} \frac{n!}{2^{n-1}}$. 
Alon’s proof

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\[
F(T) = \text{per}(A_T) \leq \prod_{i=1}^{n} (r_i !)^{1/r_i}.
\]

Here \( r_i \) is \( i \)-th row sum of \( A_T \); \( \sum_{i=1}^{n} r_i = \binom{n}{2} \).
Lemma: For every two integers $a, b$ satisfying $b \geq a + 2 > a \geq 1$, we have

$$\left( \frac{a!}{a} \right)^{1/a} \left( \frac{b!}{b} \right)^{1/b} < \left( \frac{(a + 1)!}{a+1} \right)^{1/(a+1)} \left( \frac{(b - 1)!}{b-1} \right)^{1/(b-1)}.$$
Lemma: For every two integers $a, b$ satisfying $b \geq a + 2 > a \geq 1$, we have

$$(a!)^{1/a}(b!)^{1/b} < ((a + 1)!)^{1/(a+1)}((b - 1)!)^{1/(b-1)}.$$ 

Proof: Let $f(x) = \frac{(x!)^{1/x}}{{((x+1)!)^{1/(1+x)}}}$. We need to show $f(a) < f(b - 1)$. It suffices to show $f(x - 1) < f(x)$.

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A convex inequality
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It can be proved using $x! > \left(\frac{x+1}{2}\right)^x$ for $x \geq 2$. \qed
Observe that $\sum_{i=1}^{n} (r_i!)^{1/r_i}$ achieves the maximum when all $r_i$’s are almost equal. We get

$$F(T) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2e}} n^{3/2} \frac{(n - 1)!}{2^n}.$$
Proof of theorem

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Construct a new tournament $T'$ for $T$ by adding a new vertex $v$, where the edges from $v$ to $T$ are oriented randomly and independently. Every Hamiltonian path in $T$ can be extended to a Hamiltonian cycle in $T'$ with probability $\frac{1}{4}$. We have

$$P(T) \leq \frac{1}{4} C(T') = O \left( n^{3/2} \frac{n!}{2^{n-1}} \right).$$
Independence number

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I = \{ v \in V : vw \in E \Rightarrow \sigma(v) < \sigma(w) \}.
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Let \( X_v \) be the indicator random variable for \( v \in I \).

\[
E(X_v) = \Pr(v \in I) = \frac{1}{d_v + 1}.
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\[
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**Turán Theorem**

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**Turán Theorem:** For $n = km + r$ ($0 \leq r < k$),

$$t(n, K_{k+1}) = m^2 \binom{k}{2} + rm(k - 1) + \binom{r}{2}.$$

The equality holds if and only if $G$ is the complete $k$-partite graph with equitable partitions, denoted by $G_{n,k}$.
For any $k \leq n$, let $q, r$ satisfy $n = kq + r$, $0 \leq r < k$. Let
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**Dual version of Turán Theorem:** If $G$ has $n$ vertices and $e$ edges. Then $\alpha(G) \geq k$ and the equality holds if and only if $G = \overline{G}_{n,k}$. 
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When the equality holds, $I$ is a constant. $G$ can not contain an induced $P_2$. Therefore $G = \overline{G}_{n,k}$.
Mantel (1907): $t(n, K_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. 
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Open conjectures

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Conjecture ($\$250$ for proof and $\$100$ for disproof:) Suppose $H$ is a bipartite graph. Prove or disprove that $t(n, H) = O(n^{3/2})$ if and only if $H$ does not contain a subgraph each vertex of which has degree $> 2$. 