Topic Course on Probabilistic Methods
(Week 12)
Random Graphs (II)

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Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)
Random graphs

- Supercritical regimes
- Barely Supercritical Phase
- The critical window
- Range V
- Threshold of connectivity
- Range VI
Now we consider $G(n, p)$ for $p = c/n$, with $c > 1$ constant. Let $y := y(c)$ be the positive real solution of $e^{-cy} = 1 - y$. Choose a large constant $K > 0$ and a small constant $\delta > 0$. Let $C(v)$ be the component of $G(n, p)$ containing $v$. 
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**Claim:** The probability of having any awkward component is $o(n^{-20})$. 
Proof: We will show for any awkward $t$, \( \Pr(|C(v)| = t) = o(n^{-22}) \). Note

\[
\Pr(|C(v)| = t) \leq \Pr(B(n - 1, 1 - (1 - \frac{c}{n})^t)) = t - 1.
\]
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If $t = o(n)$, then $1 - (1 - \frac{c}{n})^t \approx \frac{ct}{n}$. So the mean is about $ct$, which is not close to $t$. If $t = xn$, then

$1 - (1 - \frac{c}{n})^t \approx 1 - e^{-cx}$. Since $1 - e^{-cx} \neq x$, so the mean is not near $t$. 
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\[
\Pr\left(B(n - 1, 1 - (1 - \frac{c}{n})t\right) = O(e^{-Ct})
\]

for some constant $C$. Since $t \geq K \log n$ and $K$ large enough, this probability is $o(n^{-22})$ as required.
Let $\alpha = \Pr(C(v) \text{ is not small})$. Then

$$\alpha = \Pr(T^p_c \geq S') \approx \Pr(T^p_c = \infty) = y.$$ 

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- $\Pr(C(v) \text{ is giant}) \approx y$.
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It remains to show the giant component is unique and of size about $yn$. 
Set $p_1 = n^{-3/2}$. Let $G_1 = G(n, p_1)$, $G = G(n, p)$, and $G^+ = G \cup G_1$. Note $G^+ \sim G(n, p^+) \text{ with } p^+ = p + p_1 - pp_1$. 
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Suppose that \( G \) has two giant components \( V_1 \) and \( V_2 \). Then the probability that \( V_1 \) and \( V_2 \) is not connected after sprinkling is at most

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(1 - p_1)^{|V_1||V_2|} = o(1).
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Now $G^+$ almost surely have a component of size at least $2(y - \delta)n > (y + \delta)n$. It is an awkward component for $G^+$. Contradiction!
Sprinkling

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Since $\delta$ can be made arbitrarily small, the unique giant component has size $(1 + o(1))yn$. 
Now we consider $G(n, p)$ with $p = (1 + \epsilon)/n$ where $\epsilon = \lambda n^{-1/3}$ for $\lambda \to \infty$. This is similar to the supercritical phase with extra caution.
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The following statements hold.

- $\Pr(\exists \text{ an awkward component } ) = O(n^{-20})$.
- The escape probability $\alpha \approx y \approx 2\epsilon$.
- Sprinkling works with $p_1 = n^{-4/3}$.
Now consider $G(n, p)$ with $p = \frac{1}{n} + \lambda n^{-4/3}$ for a fixed $\lambda$. This critical window has been studied by Bollabás, Łuczak, Janson, Knuth, Pittel and many others. It requires delicate calculations.
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For fixed $c > 0$, Let $X$ be the number of tree components of size $k = cn^{2/3}$. Then

$$E(X) = \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k) + \binom{k}{2} - (k-1)}.$$
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$$

Recall

$$
\ln(1 + x) = x - \frac{1}{2}x^2 + O(x^3).
$$
We estimate

\[
\binom{n}{k} \approx \frac{n^k}{(k/e)^k \sqrt{2\pi k}} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right),
\]

and

\[
\prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) = e^{\sum_{i=1}^{k-1} \ln(1-i/n)}
\]

\[
= e^{-\sum_{i=1}^{k-1} (i/n + i^2/2n^2 + O(i^3/n^3))}
\]

\[
= e^{-\frac{k^2}{2n} - \frac{k^3}{6n^2} + o(1)}
\]

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= e^{-\frac{k^2}{2n} - \frac{c^3}{6} + o(1)}.
\]
We also estimate

\[
p^{k-1} = n^{1-k}(1 + \lambda n^{-1/3})^{k-1}
= n^{1-k} e^{(k-1) \ln(1+\lambda n^{-1/3})}
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and

\[ (1 - p)^{k(n-k) + \binom{k}{2} - (k-1)} = e^{(kn-k^2/2+O(k))\ln(1-p)} \]
\[ = e^{-(kn-k^2/2+O(k))(p+p^2/2+O(p^3))} \]
\[ = e^{-k + \frac{k^2}{2n} - \frac{\lambda k}{n^{1/3}} + \frac{\lambda c^2}{2} + o(1)} . \]
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\[ = e^{-(kn-k^2/2+O(k))(p+p^2/2+O(p^3))} \]

\[ = e^{\frac{-k^2}{2n} - \frac{\lambda k}{n^{1/3}} + \frac{\lambda c^2}{2} + o(1)}. \]
We get

$$E(X) \approx nk^{-5/2}(2\pi)^{-1/2}e^A,$$

where

$$A = \frac{(\lambda - c)^3 - \lambda^3}{6}.$$
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where
\[ A = \frac{(\lambda-c)^3-\lambda^3}{6}. \]
For any fixed \( a, b, \lambda \), let \( X \) be the number of tree components of size between \( an^{2/3} \) and \( bn^{2/3} \). Then
\[ \lim_{n \to \infty} E(X) = \int_a^b e^{A(c)}c^{-5/2}(2\pi)^{-1/2}dc. \]
Wright (1977): For a fixed $l$, there are asymptotically
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Let $X^{(l)}$ be the number of components on $k$ vertices with $k - 1 + l$ edges. Then a similar calculation shows

$$\lim_{n \to \infty} E(X^{(l)}) = \int_a^b e^{A(c)} c^{-5/2} (2\pi)^{-1/2} (c_l c^{3l/2}) dc.$$
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$$

Let $X^*$ be the total number of components of size between $an^{2/3}$ and $bn^{2/3}$. Let $g(c) = \sum_{l=0}^{\infty} c_l c^{3l/2}$. Then

$$
\lim_{n \to \infty} E(X^*) = \int_a^b e^{A(c)} c^{-5/2} (2\pi)^{-1/2} g(c) dc.
$$
For a fixed $k$, consider two random graphs $G(n, p)$ and $G(n', p')$. Assume $c = np > 1$ and $c' = n'p' < 1$. We say $G(n, p)$ and $G(n', p')$ are dual to each other if $ce^{-c} = c'e^{-c'}$. 
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Let $y = 1 - c'/c$. Then $y$ satisfies the equation $e^{-cy} = 1 - y$. Hence the size of the giant component in $G(n, p)$ is roughly $yn$. We have

$$
\lim_{n\to\infty} \Pr(C(v) = k \text{ in } G(n, p) \mid C(v) \text{ is small})
= \frac{1}{1 - y} \frac{e^{-ck}(ck)^{k-1}}{k!} = \frac{e^{-c'k}(c'k)^{k-1}}{k!}
= \lim_{n'\to\infty} \Pr(C(v) = k \text{ in } G(n', p')).
$$
Consider $G(n, p)$ with

$$p = \frac{\log n}{kn} + \frac{(k - 1) \log \log n}{kn} + \frac{t}{n} + o\left(\frac{1}{n}\right),$$

then there are only trees of size at most $k$ except for the giant component. Let $X$ be the number of trees of $k$ vertices.
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$$E(X) = \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k)} \binom{k}{2} = \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}.$$
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$$E(X) = \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k)} + \binom{k}{2}^{-k+1}
\approx \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}.$$

Further, $X$ follows the Poisson distribution.
Threshold of connectivity

For $k = 1$ and $p = \frac{\log n}{n} + \frac{t}{n} + o\left(\frac{1}{n}\right)$, $G(n, p)$ consists of a giant component with $n - O(1)$ vertices and bounded number of isolated vertices.
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- The distribution of the number of isolated vertices again has a Poisson distribution with mean value $e^{-t}$. 
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- The probability that $G(n, p)$ is connected tends to $e^{e^{-t}}$.
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- The distribution of the number of isolated vertices again has a Poisson distribution with mean value $e^{-t}$.
- The probability that $G(n, p)$ is connected tends to $e^{-e^{-t}}$.
- As $t \to \infty$, $G(n, p)$ is almost surely connected.
Consider $G(n, p)$ with $p \sim \omega(n) \log n / n$ where $\omega(n) \to \infty$. 
Consider $G(n, p)$ with $p \sim \omega(n) \log n / n$ where $\omega(n) \to \infty$. In this range, $G_{n,p}$ is not only almost surely connected, but the degrees of almost all vertices are asymptotically equal.
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Let $X = d_v$ be the degree of $v$. By Chernoff’s inequality, with probability at least $1 - O(n^{-2})$, we have

$$|X - \mathbb{E}(X)| < 2\sqrt{\omega(n) \log n}.$$ 

Almost surely, for all $v$, $d_v$ is in the interval $[d - 2\sqrt{\omega(n) \log n}, d + 2\sqrt{\omega(n) \log n}]$, where $d = np$ is the expected degree.
**Theorem:** Let $H$ be a strictly balanced graph with $v$ vertices, $m$ edges, and $a$ automorphisms. Let $c > 0$ be arbitrary. Then with $p = cn^{-v/m}$,

$$\lim_{n \to \infty} \Pr(G(n, p) \text{ contains no } H) = e^{-c^m/a}.$$
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Proof: Let $A_\alpha$, $1 \leq \alpha \leq \binom{n}{v} v!/a$, range over the edge sets of possible copies of $H$ and $B_\alpha$ be the event $A_\alpha \subset G(n, p)$. We will apply Janson’s Inequality.

$$\lim_{n \to \infty} \mu = \lim_{n \to \infty} \binom{n}{v} v! p^m / a = c^m / a.$$ 

$$\lim_{n \to \infty} M = e^{-c^m/a}.$$
Proof

Consider $\Delta = \sum_{\alpha \sim \beta} \Pr(B_\alpha \land B_\beta)$. We split the sum according to the number of vertices in $A_\alpha \cap A_\beta$. For $2 \leq j \leq v$, let $f_j$ be the maximal number of edges in $A_\alpha \cap A_\beta$ where $\alpha \sim \beta$ and $\alpha$ and $\beta$ intersect in $j$ vertices. Since $H$ is strictly balanced,

$$\frac{f_j}{j} < \frac{m}{v}.$$

There are $O(n^{2v-j})$ choices of $\alpha$, $\beta$. For such $\alpha$, $\beta$,

$$\Pr(B_\alpha \land B_\beta) = p^{|A_\alpha \cup A_\beta|} \leq p^{2m-f_j}.$$
\[ \Delta \leq \sum_{j=2}^{v} O(n^{2v-j})O(n^{(v/m)(2m-f_j)}). \]
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But

\[ 2v - j - (v/m)(2m - f_j) = \frac{vf_j}{e} - j < 0. \]
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Each term is \( o(1) \) and hence \( \Delta = o(1) \).
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Each term is \( o(1) \) and hence \( \Delta = o(1) \). By Janson’s inequality,

\[ \lim_{n \to \infty} \Pr(\bigwedge \bar{B}_\alpha) = \lim_{n \to \infty} M = e^{-c^m/a}. \]

The proof is finished.
Clique number of $G(n, \frac{1}{2})$

For the rest of slides, we assume $p = \frac{1}{2}$ and $G := G(n, 1/2)$. Let $\omega(G)$ Be the clique number. For a fixed $c > 0$, let $n, k \to \infty$ so that

$$\binom{n}{k} 2^{-\binom{k}{2}} \to c.$$
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We get $n \sim \frac{k}{e\sqrt{2}} 2^{k/2}$ and $k \sim \frac{2\ln n}{\ln 2}$. 
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For this $k$, apply Poisson paradigm to $X$: the number of $k$-cliques. We have

$$\Pr(\omega(G < k) = \Pr(X = 0) = (1 + o(1)) e^{-c}.$$
Let \( n_0(k) \) be the minimum \( n \) for which \( \binom{n}{k} 2^{-\binom{k}{2}} \geq 1 \). For any \( \lambda \in (-\infty, +\infty) \) if \( n = n_0(k) \left[ 1 + \frac{\lambda + o(1)}{k} \right] \), then

\[
\left( \binom{n}{k} \right) 2^{-\binom{k}{2}} = \left[ 1 + \frac{\lambda + o(1)}{k} \right]^k = e^\lambda + o(1).
\]

and so

\[
\Pr(\omega(G) < k) = e^{-e^\lambda} + o(1).
\]

Note that \( e^{-e^\lambda} \) ranges from 1 to 0 as \( \lambda \) ranges from \(-\infty\) to \(+\infty\). Let \( K \) be arbitrarily large and set

\[
I_k = [n_0(k)(1 - K/k), n_0(k)(1 + K/k)] .
\]
For $k \geq k_0(K)$, $I_{k-1} \cap I_k = \emptyset$ since $n_0(k + 1) \sim \sqrt{2}n_0(k)$.

If $n$ lies between the intervals, $I_k < n < I_{k+1}$, then

$$\Pr(\omega(G) = k) \geq e^{-e^{-K}} - e^{-e^K} + o(1).$$

With probability near one, we have $\omega(G) = k$. 
For \( k \geq k_0(K) \), \( I_{k-1} \cap I_k = \emptyset \) since \( n_0(k + 1) \sim \sqrt{2}n_0(k) \).

- If \( n \) lies between the intervals, \( I_k < n < I_{k+1} \), then
  \[
  \Pr(\omega(G) = k) \geq e^{-e^{-K}} - e^{-e^K} + o(1).
  \]
  With probability near one, we have \( \omega(G) = k \).

- If \( n \) lies in the interval \( I_k \), then we still have \( I_{k-1} < n < I_{k+1} \), then
  \[
  \Pr(\omega(G) = k - 1 \text{ or } k) \geq e^{-e^{-K}} - e^{-e^K} + o(1).
  \]
  With probability near one, we have \( \omega(G) = k - 1 \text{ or } k \).
Let \( f(k) = \binom{n}{k}2^{-\binom{k}{2}} \) and \( k_0 = k_0(n) \) be that value for which

\[
f(k_0 - 1) > 1 > f(k_0).
\]

Setting \( k := k_0(n) - 4 \), then \( f(k) > n^{3+o(1)} \).

We apply the Extended Janson Inquality to estimate \( \Pr(\omega(G') < k) \). We have \( \Delta \mu^2 = \sum_{i=2}^{k-1} g(i) \), where

\[
g(i) = \binom{k}{i} \binom{n-k}{k-i} \frac{i^2}{i^2} \binom{n}{k}.
\]

As \( k \sim 2 \log_2 n \), \( g(2) \sim k^4/n^2 \) dominates. Thus,

\[
\Pr(\omega(G') < k) < e^{-\mu^2/2\Delta} = e^{-\Theta(n^2/\ln^4 n)}.
\]
Theorem Bollabás (1988): Almost surely

\[ \chi(G) \sim \frac{n}{2 \log_2 n}. \]
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\[ \chi(G) \sim \frac{n}{2 \log_2 n}. \]

**Proof:** Note that \( \alpha(G) = \omega(\bar{G}) \) and \( \bar{G} \) has the same distribution as \( G(n, 1/2) \). We have \( \alpha(G) \leq (2 + o(1)) \log_2 n \). Thus almost surely
\[
\Pr(\chi(G) \geq \frac{n}{\alpha(G)}) \geq (1 + o(1)) \frac{n}{2 \log_2 n}.
\]
Let $m = \left\lfloor \frac{n}{\ln^2 n} \right\rfloor$. For any set $S$ of $m$ vertices the restriction $G|_S$ has the distribution $G(m, \frac{1}{2})$. Let $k := k(m)$ as before. Note

$$k \sim 2 \log_2 m \sim 2 \log_2 n.$$ 

There are at most $\binom{n}{m} < 2^n = 2^{m^{1+o(1)}}$ such set of $S$. Hence

$$\Pr(\exists S (\alpha(G|_S) < k)) < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1).$$
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$$\Pr(\exists S(\alpha(G|_S) < k)) < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1).$$

Almost surely every $m$ vertices contain a $k$-element independent set.
Now we pull out $k$-element independent sets and give each a distinct color until there are less than $m$ vertices left. Then we given each point a distinct color. We have

$$
\chi(G) \leq \left\lceil \frac{n - m}{k} \right\rceil + m
$$

$$
= (1 + o(1)) \frac{n}{2 \log_2 n} + o \left( \frac{n}{\log_2 n} \right)
$$

$$
= (1 + o(1)) \frac{n}{2 \log_2 n}.
$$

The proof is finished. \qed