Math778P Homework 4 Solution

1. Let \( G = (V, E) \) be a simple graph and suppose each \( v \in V \) is associated with a set \( S(v) \) of colors of size at least \( 10d \), where \( d \geq 1 \). Suppose, in addition, that for each \( v \in V \) and \( c \in S(v) \) there at most \( d \) neighbors \( u \) of \( v \) such that \( c \) lies in \( S(u) \). Prove that there is a proper coloring of \( G \) assigning to each vertex \( v \) a color from its class \( S(v) \).

**Proof by Cliff:** Without loss of generality, assume each color class has size \( 10d \). Consider a coloring of \( G \) that assigns to each vertex a color from its class. Let \( A^c_{uv} \) be the event that \( c(u) = c(v) = c \) where \( uv \in E(G) \). We create a dependency graph for this probability space. Then there are three separate types of events that share a dependency with \( A^c_{uv} \), i.e. \( A^c_{uv} \sim A^c_{u'v'} \). That is if (i) : \( u = u' \), (ii) : \( v = v' \) or (iii) : \( uv = u'v' \) and \( c \neq c' \). In case (i), we have \( 10d \) choices for \( c' \) and \( d \) choices for \( v' \) giving an upper bound of \( 10d^2 \) dependencies. The same goes for case (ii). For case (iii), we have \( 10d - 1 \) choices for \( c' \). Thus the total number of dependencies \( D \) is bounded by \( 20d^2 + 10d \).

Also \( \Pr(A^c_{uv}) = \left( \frac{1}{10d} \right)^2 \). So we may apply the Symmetric version of the Lovasz Local Lemma to see that \( e \ast \Pr(A^c_{uv}) \ast D \leq e \ast (20d^2 + 10d) \leq 100 \). Since \( d \geq 1 \), so LLL tells us that \( \Pr(\bigcap_{u,v,c} A^c_{uv}) > 0 \Rightarrow \) there exists a proper list coloring of \( G \). □

2. Let \( G = (V, E) \) be a cycle of length \( 4n \) and let \( V = V_1 \cup V_2 \cup \cdots \cup V_n \) be a partition of its \( 4n \) vertices into \( n \) pairwise disjoint subsets, each of cardinality 4. Is it true that there must be an independent set of \( G \) containing precisely one vertex from each \( V_i \)? (Prove or supply a counter example.)

**Proof by Travis:** Uniformly and independently at random select one vertex from each \( V_i \). For each \( 1 \leq i \leq n \) label the vertices in \( V_i = \{ v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} \} \). For each \( 1 \leq i < j \leq n \), let \( A_{i,k_1,j,k_2} \) be the event that \( v_{i,k_1} \) and \( v_{j,k_2} \) are chosen and that \( v_{i,k_1} \sim v_{j,k_2} \). It follows that

\[
P(A_{i,k_1,j,k_2}) = \begin{cases} 
0 & \text{if } v_{i,k_1} \text{ is not adjacent to } v_{j,k_2} \\
\frac{1}{16} & \text{if } v_{i,k_1} \text{ is adjacent to } v_{j,k_2}
\end{cases}
\]

For \( 1 \leq i < j \leq n \) and for each \( 1 \leq k_1, k_2 \leq 4 \) define:

\[
x_{i,k_1,j,k_2} = \begin{cases} 
0 & \text{if } v_{i,k_1} \text{ is not adjacent to } v_{j,k_2} \\
\frac{1}{2} & \text{if } v_{i,k_1} \text{ is adjacent to } v_{j,k_2}
\end{cases}
\]

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Let $D$ be the dependency digraph of the events $A_{i,k_1,j,k_2}$, with vertices indexed similarly to the subscripts of $A$. We now intend to show that for each event,

$$P(A_{i,k_1,j,k_2}) \leq x_{i,k_1,j,k_2} \prod_{(i,k_1,j,k_2) \sim a} (1 - x_a).$$

Note first that if $v_{i,k_1} \not\sim v_{j,k_2}$ then $P(A_{i,k_1,j,k_2}) = 0$ and $x_{i,k_1,j,k_2} = 0$, so the inequality holds for all events corresponding to non-adjacent vertices. If $v_{i,k_1} \sim v_{j,k_2}$ then $v_{i,k_1}$ only has a single additional neighbor (since its degree is 2) and likewise for $v_{j,k_2}$. Thus, there at most 2 events that are dependent on $A_{i,k_1,j,k_2}$, say events $A_a$ and $A_b$. Then $x_a = \frac{1}{2}$ and $x_b = \frac{1}{2}$. It follows that

$$P(A_{i,k_1,j,k_2}) = \frac{1}{16} \leq \frac{1}{8} = x_{i,k_1,j,k_2} \cdot (1 - x_a)(1 - x_b).$$

Thus, the condition of the Local Lemma are satisfied. Hence, with positive probability, no event $A_{i,k_1,j,k_2}$ holds; in other words, the chosen set is in fact independent.

3. Prove that there is an absolute constant $c > 0$ such that for every $k$ there is a set $S_k$ of at least $ck \ln k$ integers, such that for every coloring of the integers by $k$ colors there is an integer $x$ for which the set $x + S$ does not intersect all color classes.

**Fang Tian**, see the reference paper “Alon, Kriz, Nesetril, How to color shift hypergraphs”.

4. A family of subsets $G$ is called **intersecting** if $G_1 \cap G_2 \neq \emptyset$ for all $G_1,G_2 \in G$. Let $F_1,F_2,\ldots,F_k$ be $k$ intersecting families of subsets of $\{1,2,\ldots,n\}$. Prove that

$$\left| \bigcup_{i=1}^k F_i \right| \leq 2^n - 2^{n-k}.$$ 

**Proof by Edward:** We begin by defining, for each $i \in [k]$,

$$F_i^* = \{ F \subseteq [n] : \exists F' \supseteq F' \subset F \}$$

Now suppose $A \in F_i^*$. Then if $A \subseteq B$, we have $B \in F_i^*$ as well, since we may use the same witness set $F'$ as we did with $A$. So $F_i^*$ is monotone increasing. Thus, for any $i,j \in [k]$, by Proposition 6.3.1, it follows

$$\Pr(F_i^* \cap F_j^*) \geq \Pr(F_i^*)\Pr(\cap F_i^*)$$
By applying this result repeatedly, this implies
\[
\Pr(\bigcap_{i=1}^{k} F_i^*) \geq \prod_{i=1}^{k} \Pr(F_i^*)
\]

Now fix any \( i \in [k] \). Take a random set \( R \subset [n] \). If \( R \in F_i^* \), then necessarily \( \overline{R} \notin F_i^* \). In particular, \( \overline{R} \in \overline{F_i^*} \) and \( R \notin \overline{F_i^*} \). On the other hand, if \( R \notin F_i^* \) then this forces \( R \in F_i^* \). Further, \( \overline{R} \notin \overline{F_i^*} \) and \( \overline{R} \in F_i^* \). That is, \( R \mapsto \overline{R} \) gives a bijection between \( F_i^* \) and \( \overline{F_i^*} \). And so, \( \Pr(F_i^*) = \Pr(\overline{F_i^*}) = \frac{1}{2} \). Thus,

\[
\Pr(\bigcap_{i=1}^{k} F_i^*) \geq \left( \frac{1}{2} \right)^k
\]

Therefore, taking the complement, we obtain

\[
1 - \left( \frac{1}{2} \right)^k \geq \Pr\left( \bigcap_{i=1}^{k} F_i^* \right) = \Pr\left( \bigcup_{i=1}^{k} F_i^* \right) \geq \frac{| \bigcup_{i=1}^{k} F_i^* |}{2^n}
\]

And so, after multiplying by \( 2^n \), we arrive at the desired inequality:

\[
| \bigcup_{i=1}^{k} F_i^* | \leq 2^n - 2^{n-k}.
\]

5. Show that the probability that in the random graph \( G(2k, 1/2) \) the maximum degree is at most \( k-1 \) is at least \( 1/4^k \).

**Proof by Taylor:** Let the vertices in the random graph \( G = G(2k, 1/2) \) be enumerated \( v_1, v_2, \ldots, v_{2k} \) and let \( G \) have the property \( Q_i \) if the degree of the vertex \( v_i \) in \( G \) is at most \( k-1 \). Observe that each \( Q_i \) is monotone decreasing, since if \( G \in Q_i \) then for any subgraph \( H \subset G \) we have \( \deg_H(v_i) \leq \deg_G(v_i) \leq k-1 \) and similarly we can show the intersection of any number of the \( Q_i \) is monotone decreasing. Since
the degree of any vertex in $G$ is at most $2k - 1$, we have

$$\Pr[G \in Q_i] = \sum_{i=0}^{k-1} \binom{2k-1}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{2k-1-i}$$

$$= \frac{1}{2} \sum_{i=0}^{2k-1} \binom{2k-1}{i} \left(\frac{1}{2}\right)^{2k-1}$$

$$= \frac{1}{2} \cdot 2^{2k-1} \cdot \left(\frac{1}{2}\right)^{2k-1}$$

$$= \frac{1}{2}.$$

Now applying theorem 6.3.3 $2k$ times we obtain

$$\Pr \left[ G \in \bigcap_{i=1}^{2k} Q_i \right] \geq \prod_{i=1}^{2k} \Pr[G \in Q_i]$$

$$\geq \left(\frac{1}{2}\right)^{2k}$$

$$= \frac{1}{4^k}$$

and therefore, the probability that in the random graph $G$ the maximum degree is at most $k - 1$ is at least $1/4^k$. \qed

6. Suppose that $p = \frac{c \ln n}{n}$, where $c > 2$ is a constant. Prove that there are two positive constants $c_1$ and $c_2$ so that with probability $1 - o_n(1)$, all degrees of the random graph $G(n, p)$ are in the interval $[c_1 \ln n, c_2 \ln n]$. Also show that the statement does not hold for $c = 1$.

**Proof by Heather:** Suppose that $p = \frac{c \ln n}{n}$, where $c > 2$ is a constant. Prove that there are two positive constants $c_1$ and $c_2$ so that with probability $1 - o_n(1)$, all degrees of the random graph $G(n, p)$ are in the interval $[c_1 \ln n, c_2 \ln n]$. Also show that the statement does not hold for $c = 1$.

Label the vertices $\{1, 2, \ldots, n\}$ in $G = G(n, p)$. Fix a vertex $i$. For each $j \neq i$, Let $X^i_j$ be the event that $ij \in E(G)$. Then $X^i = \sum_{j \neq i} X^i_j$ is the degree of vertex $i$ in $G$. Observe

$$\mathbb{E}(X^i) = (n-1)p = \frac{c(n-1) \ln n}{n}.$$
Since \( \Pr(X_j = 1) = p \) and \( \Pr(X_j = 0) = 1 - p \), the Chernoff inequality,

\[
\Pr(X^i < E(X^i) - \lambda) \leq e^{-\frac{\lambda^2}{2E(X^i)}}.
\]

First observe \( \left( \frac{1}{2} + \frac{1}{c} \right)^{1/2} < 1 \) since \( c > 2 \). Pick \( s \in \left( \left( \frac{1}{2} + \frac{1}{c} \right)^{1/2}, 1 \right) \).

Let \( \gamma := \left( \frac{c^2}{2} + c \right)^{1/2} \) Let \( c_1 := sc - \gamma \) which is positive by the choice of \( s \). Observe

\[
\Pr(X^i < c_1 \ln n) = \Pr(X^i < sc \ln n - \gamma \ln n)
\]

\[
\leq \Pr\left( X^i < \frac{n-1}{n} \ln n - \gamma \ln n \right)
\]

for all \( n \) suff. large so that \( \frac{n-1}{n} > s \)

\[
\leq e^{-\frac{c^2 (n-1) \ln n}{2c(n-1) \ln n}}
\]

\[
\leq e^{-\frac{\gamma^2 \ln n}{2c}}
\]

\[
= e^{-\frac{(\frac{c^2}{2} + c) \ln n}{2}}
\]

\[
= n^{-\frac{c}{4} - \frac{1}{2}}.
\]

Let \( A_i \) be the event that vertex \( i \) has degree less than \( c_1 \ln n \). Therefore, the probability that all vertices have degree at least \( c_1 \ln n \) is equal to \( \Pr(\bigwedge_{i=1}^n \overline{A_i}) \). Additionally, notice that each \( A_i \) is a monotone decreasing event. Therefore, by the FKG inequality:

\[
\Pr\left( \bigwedge_{i=1}^n \overline{A_i} \right) \geq \prod_{i=1}^n \Pr(\overline{A_i})
\]

\[
= \prod_{i=1}^n \left( 1 - \Pr(A_i) \right)
\]

\[
\geq \left( 1 - n^{-\frac{c}{4} - \frac{1}{2}} \right)^n
\]

\[
= \left( 1 - \frac{1}{n^{\frac{c}{4} + \frac{1}{2}}} \right)^{n^{\frac{c}{4} + \frac{1}{2} + \frac{1}{2} - \frac{c}{4}}}
\]

\[
e^{-n^{\frac{1}{2} - \frac{c}{4}}}
\]

\[
= \frac{1}{e^{n^{\frac{1}{2} - \frac{c}{4}}}}
\]

\[
\rightarrow 1 \quad \text{since} \quad \frac{1}{2} - \frac{c}{4} < 0
\]
Therefore the probability that all vertices have degree at least \( c_1 \ln n \) is at least \( 1 - o(1) \). Equivalently, the probability that some vertex has degree less than \( c_1 \ln n \) is \( o(1) \).

For the other bound, define \( d := \frac{1+\sqrt{1+18c_3}}{3} \). Let \( c_2 := c + d > 0 \).

Consider the following:

\[
\Pr(X^i > c_2 \ln n) = \Pr\left(X^i > c \ln n - d \ln n\right)
\leq \Pr\left(X^i > \frac{n-1}{n}c \ln n + d \ln n\right) \quad \text{since } \frac{n-1}{n} < 1
\leq e^{-\frac{d^2(n-1)c \ln n + d \ln n}{2n^2}}
\leq e^{-\frac{d^2 \ln n}{n(2c+4d)}}
= n^{-\frac{2ad^2}{(n-1)c + 2ad^2}}.
\]

Let \( B_i \) be the event that vertex \( i \) has degree greater than \( c_2 \ln n \). If we can show \( \Pr\left(\bigwedge_{i=1}^n B_i\right) \geq 1 - o(1) \) then \( \Pr\left(\bigvee_{i=1}^n B_i\right) \leq o(1) \). So the probability that all vertices have degree in \([c_1 \ln n, c_2 \ln n]\) is

\[
\Pr\left(\bigwedge_{i=1}^n \overline{A_i} \land \bigwedge_{i=1}^n B_i\right) = 1 - \Pr\left(\bigvee_{i=1}^n A_i \lor \bigvee_{i=1}^n B_i\right)
\geq 1 - \Pr\left(\bigvee_{i=1}^n A_i\right) - \Pr\left(\bigvee_{i=1}^n B_i\right)
\geq 1 - o(1).
\]

\(\square\)