Math 550, Exam 1, solution, Spring 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don’t worry about it – I will still grade your exam.

The exam is worth 50 points. **SHOW your work.** *CIRCLE* your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

The solutions will be posted later today.

1. (9 points) *Compute* \( \int_0^1 \int_y^1 \sin(x^2) \, dx \, dy \). **Explain very carefully what you are doing.**

None of us know an elementary anti-derivative for \( \sin(x^2) \). Let’s see if things get better after we exchange the order of integration. A picture is available on a different page. The original integral is equal to

\[
\int_0^1 \int_0^x \sin(x^2) \, dy \, dx = \int_0^1 \sin(x^2) y \, dx = \int_0^1 \sin(x^2) x \, dx = -\frac{\cos(x^2)}{2} \bigg|_0^1 = \frac{1}{2} (1 - \cos(1)).
\]

2. (9 points) Let \( f(x) \) be a continuous function for \( a \leq x \leq b \). **Find a formula which relates** \( (\int_a^b f(x) \, dx)^2 \) **and** \( \int_a^b \int_a^b f(x) f(y) \, dy \, dx \). **Explain why your formula is correct very carefully.**

We see that

\[
(\int_a^b f(x) \, dx)^2 =_1 (\int_a^b f(x) \, dx)(\int_a^b f(x) \, dx) =_2 (\int_a^b f(x) \, dx)(\int_a^b f(y) \, dy)
\]

\[
= _3 (\int_a^b f(x)(\int_a^b f(y) \, dy) \, dx) =_4 (\int_a^b (\int_a^b f(x)f(y) \, dy) \, dx)
\]

\[
= _5 \int \int_{[a,b] \times [a,b]} f(x)f(y) \, dA.
\]

The equalities 1 and 2 are obvious. For equality 3, it is legal to move the constant \( \int_a^b f(y) \, dy \) inside the integral \( \int_a^b f(x) \, dx \). For equality 4, as far as the integral \( \int_a^b f(y) \, dy \) is concerned, \( f(x) \) is a constant. It is legal to move the constant inside the integral sign. The left side of equality 4 is an iterated integral; the right side is the corresponding double integral. We split the rectangle \( [a,b] \times [a,b] \) into two
triangles by drawing the line connecting the corner \( (a, a) \) to the corner \( (b, b) \). (I put a picture on a different page.)

\[
= 6 \left\{ \int \int \text{the triangle with vertices } (a, a), (a, b), (b, b) \right. f(x) f(y) \, dA \\
+ \left. \int \int \text{the triangle with vertices } (a, a), (b, a), (b, b) \right. f(x) f(y) \, dA \\
\]

We fill up the triangle of the first integral using vertical lines. We fill up the triangle of the second integral using horizontal lines.

\[
= 7 \int_{a}^{b} \int_{x}^{b} f(x) f(y) \, dy \, dx + \int_{a}^{b} \int_{y}^{b} f(x) f(y) \, dx \, dy \\
= 8 \int_{a}^{b} \int_{x}^{b} f(x) f(y) \, dy \, dx + \int_{a}^{b} \int_{y}^{b} f(x) f(y) \, dx \, dy = 2 \int_{a}^{b} \int_{x}^{b} f(x) f(y) \, dy \, dx.
\]

In 8, we replaced all the \( x \)'s by \( y \)'s and all of the \( y \)'s by \( x \)'s in the second integral.

We have shown that

\[
\left( \int_{a}^{b} f(x) \, dx \right)^{2} = 2 \int_{a}^{b} \int_{x}^{b} f(x) f(y) \, dy \, dx
\]

3. (8 points) A lumberjack cuts a wedge-shaped piece \( W \) out of a cylindrical tree of radius \( a \) by making two saw cuts. The first cut is parallel to the ground. The second cut makes an angle \( \theta \) with the first cut and meets the first cut along a diagonal of the circle that contains the first cut. Find the volume of \( W \). Explain very carefully what you are doing.

We use a triple integral. The outer two integrals are over the base. The inner integral is from the bottom (\( z = 0 \)) to the top (\( z = x \tan \theta \)). The base is the semi-circle with positive \( x \) and inside \( x^2 + y^2 = a^2 \). I drew a picture elsewhere. The volume of \( W \) is

\[
\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x \tan \theta \, dz \, dy = \int_{-a}^{a} \int_{0}^{x \tan \theta} \, dx \, dy \\
\]

\[
= \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} x \tan \theta \, dy = \int_{-a}^{a} \frac{x^2}{2} \tan \theta \left|_{0}^{\sqrt{a^2-y^2}} \right. \, dy = \int_{-a}^{a} \frac{a^2 - y^2}{2} \tan \theta \, dy \\
= \left( \frac{a^2 y}{2} - \frac{y^3}{6} \right) \tan \theta \bigg|_{-a}^{a} = 2 \left( \frac{a^3}{2} - \frac{a^3}{6} \right) \tan \theta = \frac{2a^3 \tan \theta}{3}
\]
4. (8 points) Let \( f(x, y, z) \) be a continuous function which is defined on all of three space. Let \( a, b, \) and \( c \) be constants. Consider the function \( F(x) = \int_{c}^{x} \int_{a}^{b} f(x, y, z) dydz \). Find an expression for \( \frac{d}{dx} F(x) \) in which all differentiation is done before all integration. Explain very carefully what you are doing.

We use the chain rule. View \( F \) as a function of \( u \) and \( v \), where \( u(x) = x \) and \( v(x) = x \) and \( F(u, v) = \int_{c}^{u} \int_{a}^{b} f(v, y, z) dydz \). The chain rule is \( \frac{d}{dx} F(x) = \frac{\partial F}{\partial u} \frac{du}{dx} + \frac{\partial F}{\partial v} \frac{dv}{dx} \). It is clear that \( \frac{du}{dx} = \frac{dv}{dx} = 1 \). To compute \( \frac{\partial F}{\partial u} \) we use the Fundamental Theorem of Calculus which says that \( \frac{d}{du} \int_{c}^{u} g(z) dz = g(u) \). For us, \( g(z) \) is the function \( \int_{a}^{b} f(v, y, z) dy \), where \( v \) is a constant as far is the calculation \( \frac{\partial F}{\partial u} \) is concerned. So \( \frac{\partial F}{\partial u} = \int_{a}^{b} f(v, y, u) dy \). To compute \( \frac{\partial F}{\partial v} \) we differentiate under the integral sign twice:

\[
\frac{\partial F}{\partial v} = \frac{\partial}{\partial v} \int_{c}^{u} \int_{a}^{b} f(v, y, z) dydz = \int_{c}^{u} \frac{\partial}{\partial v} \int_{a}^{b} f(v, y, z) dydz
\]

\[
= \int_{c}^{u} \int_{a}^{b} \frac{\partial}{\partial v} f(v, y, z) dydz = \int_{c}^{u} \int_{a}^{b} f_{v}(v, y, z) dydz.
\]

We have shown that

\[
\frac{d}{dx} F(x) = \frac{\partial F}{\partial u} \frac{du}{dx} + \frac{\partial F}{\partial v} \frac{dv}{dx}
\]

\[
= \int_{a}^{b} f(v, y, u) dy + \int_{c}^{u} \int_{a}^{b} f_{v}(v, y, z) dydz = \int_{a}^{b} f(x, y, x) dy + \int_{c}^{u} \int_{a}^{b} f_{x}(x, y, z) dydz.
\]

We conclude

\[
\frac{d}{dx} F(x) = \int_{a}^{b} f(x, y, x) dy + \int_{c}^{u} \int_{a}^{b} f_{x}(x, y, z) dydz.
\]

5. (8 points) Find a linear map \( L: \mathbb{R}^2 \to \mathbb{R}^2 \) which carries the parallelogram with vertices \((0,0),(a,b),(c,d),(a+c,b+d)\) to the parallelogram with vertices \((0,0),(e,f),(g,h),(e+g,f+h)\). (You may assume that both parallelograms are honest-to-goodness parallelograms.) Explain very carefully what you are doing.

We take \( L \) to be the transformation \( L = S \circ T^{-1} \) where \( T \) is the transformation that carries the unit square to the parallelogram with vertices \((0,0),(a,b),(c,d),(a+c,b+d)\) and \( S \) is the transformation that carries the unit square to parallelogram with vertices \((0,0),(e,f),(g,h),(e+g,f+h)\). Thus

\[
T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad S \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},
\]
and
\[
L \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{ad - bc} \begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ed - gb \\ fd - hb \end{bmatrix} \begin{bmatrix} ga - ec \\ ha - fc \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

We conclude that
\[
L \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{ad - bc} \begin{bmatrix} ed - gb \\ fd - hb \end{bmatrix} \begin{bmatrix} ga - ec \\ ha - fc \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

6. (8 points) **What is the area of the parallelogram with vertices \((0, 0), (a, b), (c, d), (a + c, b + d)\)?** (You may assume that the parallelogram is an honest-to-goodness parallelogram.) Explain very carefully what you are doing.

We calculated in class that the area is \(|\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}| = |ad - bc|\). Our argument went something like this. Let \(v\) be the vector \(\begin{bmatrix} a \\ b \end{bmatrix}\) and \(w\) be the vector \(\begin{bmatrix} c \\ d \end{bmatrix}\).

The area of the parallelogram determined by \(v\) and \(w\) is the length of the base time the height. We take \(v\) to be the base. Then the height is the length of \(w\) minus the projection of \(w\) onto \(v\). (I have drawn a picture.) The area is
\[
||v|| \| (w - \text{proj}_v w) || = ||v|| \| (w - \frac{v \cdot w}{v \cdot v} v) || = \sqrt{(v \cdot v)(w \cdot w - \frac{v \cdot w}{v \cdot v} v \cdot w + \left(\frac{v \cdot w}{v \cdot v}\right)^2 v \cdot v)}
\]
\[
= \sqrt{(v \cdot v)(w \cdot w) - 2v \cdot w + (v \cdot w)^2}
\]
\[
= \sqrt{(v \cdot v)(w \cdot w) - (v \cdot w)^2}
\]
\[
= \sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2}
\]
\[
= \sqrt{(a^2 c^2 + a^2 d^2 + b^2 c^2 + b^2 d^2) - (a^2 c^2 + 2abcd + b^2 d^2)}
\]
\[
= \sqrt{a^2 d^2 + b^2 c^2 - 2abcd}
\]
\[
= \sqrt{(ad - bc)^2} = |ad - bc|
\]