INSTRUCTIONS:
(1) Do Problem 0. Do 3 of the problems: 1, 2, 3, 4, and 5.
(2) Use complete sentences.
(3) When asked to show, this really means carefully show. Write out details.
(4) Write your solutions on the provided unlined or lined homemade Blue Book paper. You do not need to recopy the statement of the problem (just use LTGBG).
   • Start each new problem on a new sheet of paper.
   • Write on only one side of a sheet of paper.
   • On the top of each page put the problem number and your (first is enough) name.
(5) When finished:
   • put your pages in order with this exam paper on top
   • staple them together
   • in the MARK BOX above, circle the problem that you chose to do.
   Then hand in your completed homemade Blue Book.
(6) The mark box indicates the problems along with their points. This test is copied 2-sided.
(7) You may not use any electronic devices, books, or personal notes.

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Honor Code Statement
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the above Instructions.

Signature: ________________________________
0a. State Cauchy’s Theorem for starlike sets. (Thm. II.2.12)

I’ll get you started. Let $G$ be an open starlike subset of $\mathbb{C}$ and $p \in G$. Let $f : G \to \mathbb{C}$ be so that $f$ and $\partial G$ are starlike. Then there exists $F \in H(G)$ so that $f = F'$. Thus for each closed contour $\gamma$ in $G$, $\int \gamma f(z) \, dz = 0$.

0b. State Cauchy’s Integral Formula for starlike sets. (Thm. II.2.14)

You do not have to define the index of a closed contour.

Solution. Let $G$ be an open starlike subset of $\mathbb{C}$ and $f \in H(G)$. Let $\gamma$ be a closed contour (i.e., closed piecewise smooth curve) such that $\gamma^* \subset G$. Let $a \in G \setminus \gamma^*$. Then

$$[f(a)] \, [\text{Ind}_\gamma(a)] = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz.$$  

0c. Let $f \in H(G)$ where $G$ is an open subset of $\mathbb{C}$. Let $B_R(a) \subset G$. Fix $0 < r < R$ and define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = a + re^{it}$. Let $n \in \mathbb{N} \cup \{0\}$. Express $f^{(n)}(a)$ as a contour integral around $\gamma$.

Hint: this expression can be thought of as an extension of Cauchy’s integral formula. By express I mean with an = sign, not a $\leq$ sign. (Cor. II.2.24 as given in class)

Solution.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} \, dz.$$  

0d. State Morera’s Theorem. (Thm. II.2.25)

Solution. Let $G$ be an open subset of $\mathbb{C}$. Let $f \in C(G)$ be such that $\int_{\partial \triangle} f(z) \, dz = 0$ for each triangle $\triangle$ contained in $G$. Then $f \in H(G)$.

0e. State Liouville’s Theorem. (Thm. II.2.26)

Solution. A bounded entire function is constant.

0f. Find (all) square roots of $1 - i\sqrt{3}$. Express your solutions in the form of $a + ib$ with $a, b \in \mathbb{R}$.

Solution. $\pm \left( \sqrt{\frac{3}{2}} + i\frac{\sqrt{3}}{2} \right)$

Details. First see comments and solution to HMWK 22. Note $|1 - i\sqrt{3}| = 2$ and so

$$1 - i\sqrt{3} = 2 \left( \frac{1}{2} - i\frac{\sqrt{3}}{2} \right) = 2e^{i\left( \frac{\pi}{3} + 2\pi k \right)}, \; k \in \mathbb{Z}.$$  

So

$$\left( 1 - i\sqrt{3} \right)^{\frac{1}{2}} = \sqrt{2} e^{i\left( \frac{\pi}{6} + \pi k \right)}, \; k \in \mathbb{Z} \quad \text{and} \quad k = 0, 1$$  

$$= \pm \sqrt{2} e^{i\frac{\pi}{6}}$$  

$$= \pm \sqrt{2} \left( \frac{\sqrt{3}}{2} - i\frac{1}{2} \right)$$  

$$= \pm \frac{\sqrt{2}}{\sqrt{2}} \left( \frac{\sqrt{3}}{2} - i\frac{1}{2} \right).$$
0g. Find (all) solutions to the equation \( z^2 + 2z + (1 - i) = 0 \). Express in the form of \( a + ib \) with \( a, b \in \mathbb{R} \).

**Solution.** \( \left( \frac{-2 + \sqrt{2}}{2} \right) + i \frac{\sqrt{2}}{2} \) and \( \left( \frac{-2 - \sqrt{2}}{2} \right) + i \frac{-\sqrt{2}}{2} \)

Details. First see comments and solution to HMWK 22.

\[ b^2 - 4ac = 4 - 4(1 - i) = 4i = 4e^{i\left( \frac{\pi}{2} + 2\pi k \right)}, \quad k \in \mathbb{Z} \]

\[ \sqrt{b^2 - 4ac} = 2e^{i\left( \frac{\pi}{4} + \pi k \right)}, \quad k \in \mathbb{Z} \] or equiv. \( k = 0, 1 \)

\[ \sqrt{b^2 - 4ac} = \pm \sqrt{2}(1 + i) \]

So the two solutions are

\[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2}(1 + i)}{2} \]

0h. Find (all) solutions to the equation \( e^z = 1 + i\sqrt{3} \). Express in the form of \( a + ib \) with \( a, b \in \mathbb{R} \).

**Solution.** \( \ln 2 + i \left( \frac{\pi}{3} + 2\pi k \right) \), \( k \in \mathbb{Z} \)

Details. First see solution to HMWK 25 (a). Recall

\[ e^z = w \iff \left[ z = \ln|w| + i(\text{Arg } w + 2\pi k), \quad k \in \mathbb{Z} \right] \]

Let’s do without this recall but with “common sense” instead. Note

\[ 1 + i\sqrt{3} = 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 2e^{i\frac{\pi}{3}} = 2e^{i\left( \frac{\pi}{3} + 2\pi k \right)}, \quad k \in \mathbb{Z} \]

Let \( z = x + iy \) with \( x, y \in \mathbb{R} \). Then \( e^z = e^x e^{iy} \). So we want

\[ e^x e^{iy} = 2e^{i\left( \frac{\pi}{3} + 2\pi k \right)}, \quad k \in \mathbb{Z} \]

i.e., we need \( x = \ln 2 \) and \( y = \left( \frac{\pi}{3} + 2\pi k \right) \) with \( k \in \mathbb{Z} \).

0i. Find (all) solutions to the equation \( \cos z = 2 \). Express in the form of \( a + ib \) with \( a, b \in \mathbb{R} \).

**Solution.** \( 2\pi k \pm \ln \left( 2 + \sqrt{3} \right) \) or \( 2\pi k + i \cosh^{-1} 2 \), \( k \in \mathbb{Z} \)

Details. First see remarks and solutions to HMWK 24.

Let \( z = x + iy \) where \( x, y \in \mathbb{R} \). By definition

\[ \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

Thus, TFAE.

\[ \cos z = 2 \]

\[ e^{iz} + e^{-iz} = 4 \]

\[ e^{iz} - 4 + e^{-iz} = 0 \]

\[ \left( e^{iz} \right)^2 - 4(e^{iz}) + 1 = 0 \]

\[ e^{iz} = \frac{4 \pm \sqrt{16 - 4}}{2} \]

\[ e^{iz} = 2 \pm \sqrt{3} \]

\[ e^{-y}e^{ix} = \left( 2 \pm \sqrt{3} \right) e^{i(0 + 2\pi k)}, \quad k \in \mathbb{Z} \]

\[ y = -\ln \left( 2 \pm \sqrt{3} \right) \] and \( x = 0 + 2\pi k \), \( k \in \mathbb{Z} \).

Note that \( (2 + \sqrt{3})^{-1} = 2 - \sqrt{3} \).
0j. Find (all) solutions to the equation \( \log z = i \left( \frac{\pi}{2} \right) \). Express in the form of \( a + ib \) with \( a, b \in \mathbb{R} \).

**Solution.** \( z = i \)

Details. From Class Script, p. 3, we know that, for \( z \in \mathbb{C} \setminus \{0\} \),
\[
\log z = w \quad \iff \quad w \in \{ \ln |z| + i (\text{Arg}(z) + 2\pi k) : k \in \mathbb{Z} \}.
\]
Thus TFAE.
\[
\begin{align*}
\log z &= 0 + i \frac{\pi}{2} \\
\ln |z| &= 0 \quad \text{and} \quad \frac{\pi}{2} \in \{ \text{Arg}(z) + 2\pi k : k \in \mathbb{Z} \} \\
|z| &= 1 \quad \text{and} \quad \text{Arg}(z) = \frac{\pi}{2} \\
z &= i.
\end{align*}
\]

1a. State the Cauchy Riemann (CR) equations.
1b. Let \( f: G \to \mathbb{C} \) be a function where \( G \) is an open subset of \( \mathbb{C} \). Let \( z \in G \). What is the relation between \( f \) being differentiable at \( z \) and the CR equations being satisfied at \( z \)? I’m looking for a \( \iff \) and a \( \implies \), with possibly more hypothesis that need to be added to make an implication valid.
1c. Define \( f: \mathbb{C} \to \mathbb{C} \) and \( u, v: \mathbb{R}^2 \to \mathbb{R} \) by:
\[
u(x, y) := \Re f(x + iy) \quad \text{and} \quad v(x, y) := \Im f(x + iy) \quad \text{and} \quad f(z) := \sqrt{|xy|} \quad \text{where} \quad x := \Re z, \quad y := \Im z.
\]
Show that
1. \( u \) and \( v \) satisfies the Cauchy Riemann equations at \( (x, y) = (0, 0) \)
2. \( f \) is not differentiable at \( z = 0 \).

**Answer.** See comments and solutions to HMWKs 26 and 27.

2. Let \( f \in H(\mathbb{C}) \) satisfy, for some constants \( A, B \in \mathbb{R} \) and \( k \in \mathbb{N} \), the inequality \( |f(z)| \leq A |z|^k + B \) for each \( z \in \mathbb{C} \). Prove that \( f \) is a polynomial.

**Answer.** See HMWK 39.

3. Let \( G \) be an open connected subset of \( \mathbb{C} \). Let \( \{f_n\}_{n=1}^\infty \) be a sequence from \( H(G) \) and \( f: G \to \mathbb{C} \) be a function satisfying that, for each compact subset \( K \) of \( G \), the functions \( \{f_n|_K\}_{n=1}^\infty \) converge uniformly on \( K \) to \( f|_K \). Show that \( f \in H(G) \).

**Answer.** See HMWK 40.
4a. Let $u : G \rightarrow \mathbb{R}$ where $G$ be an open subset of $\mathbb{R}^2$. Define what it means for $u$ to be harmonic of $G$.

4b. Show that $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $u(x, y) = x^3 - 3xy^2 + y$ is harmonic.

4c. Construct an entire function $f$ whose real part is $u(x, y) = x^3 - 3xy^2 + y$.

Answer. (a) $u$ is harmonic on $G$ provided:

1. the first and second order partial derivatives of $u$ exist and are continuous of $G$
2. $u$ satisfies the Laplace equation on $G$, i.e.,
   $$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ 

(b) and (c). This is an example from class so see your notes. Note that

$$
\begin{align*}
  u_x &= 3x^2 - 3y^2 & u_y &= -6xy + 1 \\
  u_{xx} &= 6x & u_{yy} &= -6x \\
  \frac{\partial u}{\partial y} &= -6y & \frac{\partial u}{\partial x} &= -6y .
\end{align*}
$$

Also, the harmonic conjugate of $u$ is

$$v(x, y) = 3x^2y - y^3 - x + C$$

so

$$f(x + iy) = (x^3 - 3xy^2 + y) + i \left(3x^2y - y^3 - x + C \right).$$

5. Give 2 different proofs of the Fundamental Theorem of Algebra that are based primarily on theory from Complex Analysis. Explain clearly, as always.

Solution. Here is a list of possible proofs.

1. Thm. II.2.16, page 21 of our Class Script. Uses the Cauchy Integral Formula.

2. The standard proof using Liouville’s Theorem, as given in class.