**INTRO to TOPOLOGY**

**Neighborhood (NBHD)**  

Setup:  

\[ S, G, F, K \subseteq \mathbb{R} \text{ and } \varepsilon > 0 \text{ and } x, y \in \mathbb{R} \]

\[
N_\varepsilon(x_0) \overset{\text{NTN}}{=} \text{\$-
NBHD of } x_0 \overset{\text{def}}{=} \{ y \in \mathbb{R} : |x_0 - y| < \varepsilon \} = (x_0 - \varepsilon, x_0 + \varepsilon) \\
N'_\varepsilon(x_0) \overset{\text{NTN}}{=} \text{deleted } \varepsilon\text{-NBHD of } x_0 \overset{\text{def}}{=} \{ y \in \mathbb{R} : 0 < |x_0 - y| < \varepsilon \} = N_\varepsilon(x_0) \setminus \{x_0\} = (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon)
\]

\[ S \text{ is a NBHD of } x_0 \overset{\text{def}}{=} \exists \varepsilon > 0 \text{ s.t. } N_\varepsilon(x_0) \subseteq S \]

**DEFINITIONS AND NOTATION**

\[ x_0 \text{ is an interior point of } S \overset{\text{NTN}}{\iff} x_0 \in S^o \overset{\text{def}}{=} \exists \varepsilon > 0 \text{ s.t. } N_\varepsilon(x_0) \subseteq S \]

\[ x_0 \text{ is a limit point of } S \overset{\text{NTN}}{\iff} x_0 \in S' \overset{\text{def}}{=} \forall \varepsilon > 0 : N_\varepsilon(x_0) \cap S \neq \emptyset \]

\[ x_0 \text{ is a boundary point of } S \overset{\text{NTN}}{\iff} x_0 \in \partial S \overset{\text{def}}{=} \forall \varepsilon > 0 : N_\varepsilon(x_0) \cap S \neq \emptyset \text{ and } N_\varepsilon(x_0) \cap S^C \neq \emptyset \]

\[ x_0 \text{ is an isolated point of } S \overset{\text{def}}{=} \{x_0 \in S\} \text{ and } \exists \varepsilon > 0 \text{ s.t. } N_\varepsilon(x_0) \cap S = \emptyset \iff \exists \varepsilon > 0 \text{ s.t. } N_\varepsilon(x_0) \cap S = \{x_0\} \]

\[ x_0 \text{ is an exterior point to } S \overset{\text{def}}{=} x_0 \in (S^C)^o \]

\[ \text{the boundary of } S \overset{\text{NTN}}{=} \partial S \overset{\text{def}}{=} \text{the set of all boundary points of } S \]

\[ \text{the closure of } S \overset{\text{NTN}}{=} \overline{S} \overset{\text{def}}{=} S \cup \partial S \]

\[ \text{the interior of } S \overset{\text{NTN}}{=} S^o \overset{\text{def}}{=} \text{the set of all interior points of } S \]

\[ \text{the exterior of } S \overset{\text{NTN}}{=} (S^C)^o \overset{\text{def}}{=} \text{the interior of the complement of } S \]

**OPEN and CLOSED**

\[ G \text{ is open } \overset{\text{def}}{=} \text{each point in } G \text{ is an interior point of } G \iff \forall x \in G \exists \varepsilon > 0 \text{ s.t. } N_\varepsilon(x) \subseteq G \]

\[ F \text{ is closed } \overset{\text{def}}{=} F^c \overset{\text{def}}{=} \mathbb{R} \setminus F \text{ is an open set} \]

**THEOREMS**

Let \( \Gamma \) be an arbitrary indexing set and \( n \in \mathbb{N} \).

**OPEN:**  

Let \( \{G_\gamma\}_{\gamma \in \Gamma} \) and \( \{G_i\}_{i=1}^n \) be collections of open subsets of \( \mathbb{R} \). Then:

\[
\bigcup_{\gamma \in \Gamma} G_\gamma \text{ is open and } \bigcap_{i=1}^n G_i \text{ is open}.
\]

**CLOSED:**  

Let \( \{F_\gamma\}_{\gamma \in \Gamma} \) and \( \{F_i\}_{i=1}^n \) be collections of closed subsets of \( \mathbb{R} \). Then:

\[
\bigcap_{\gamma \in \Gamma} F_\gamma \text{ is closed and } \bigcup_{i=1}^n F_i \text{ is closed}.
\]

**RECALL:**  

\[
x \in \bigcup_{\gamma \in \Gamma} G_\gamma \overset{\text{def}}{\iff} \exists \gamma \in \Gamma \text{ s.t. } x \in G_\gamma
\]

\[
x \in \bigcap_{\gamma \in \Gamma} F_\gamma \overset{\text{def}}{\iff} \forall \gamma \in \Gamma \text{ : } x \in F_\gamma
\]

**MORE THEOREMS**

- Let \( x \in S \). Then \( x \) is either an isolated point or a limit point of \( S \) (but not both).  
- \( S \) is closed \( \iff \) \( S \) contains all its limit points \( \iff \) \( S^c \subseteq S \).
(1) interior of \( S \) \( \overset{\text{NTN}}{=} S^o \) def. = set of interior points of \( S \)
(2) \( S^o \) is open
(3) \( S^o \subset S \)
(4) \( S^o = S \iff S \) is open
(5) \( S^o = \bigcup \{G \subset \mathbb{R} : G \text{ is open and } G \subset S\} \)
= the largest open set “inside of” \( S \)
= the largest open set contained in \( S \)

(1) closure of \( S \) \( \overset{\text{NTN}}{=} \overline{S} \) def. = \( S \cup \partial S \) \( \overset{\text{thm}}{=} S \cup S' \)
(2) \( \overline{S} \) is closed
(3) \( S \subset \overline{S} \)
(4) \( S = \overline{S} \iff S \) is closed
(5) \( \overline{S} = \bigcap \{F \subset \mathbb{R} : F \text{ is closed and } S \subset F\} \)
= the smallest closed set that “sits on top of” \( S \)
= the smallest closed set that contains \( S \)

**COMPACT SETS**

• A collection

\[ \mathcal{C} = \{G_\gamma\}_{\gamma \in \Gamma} \]

of subsets of \( \mathbb{R} \) is an OPEN COVERING of \( S \) if each \( G_\gamma \) is open and the \( G_\gamma \)'s cover \( S \) in the sense that

\[ S \subset \bigcup_{\gamma \in \Gamma} G_\gamma . \]

If

\[ \tilde{\mathcal{C}} = \{G_\gamma\}_{i=1}^n \subset \mathcal{C} , \]

where \( n \in \mathbb{N} \), and

\[ S \subset \bigcup_{i=1}^n G_{\gamma_i} \]

then \( \tilde{\mathcal{C}} \) is a FINITE SUBCOVERING of \( S \) (of the covering \( \mathcal{C} \)).

• \( K \) is COMPACT \(^9\) if each open covering of \( K \) has a finite subcovering of \( K \). So:

\[ K \text{ is compact } \iff \forall \text{ open covering } \mathcal{C} \text{ of } K \exists \text{ finite subcovering } \tilde{\mathcal{C}} \text{ of } K . \]

**HEINE-BOREL THEOREM** \(^{10}\)

Let \( S \subset \mathbb{R} \).

Each open covering of \( S \) has a finite subcovering if and only if \( S \) is closed and bounded.

In other words:

\[ S \text{ is compact } \iff S \text{ is closed and bounded} . \]

**BOLZANO-WEIERSTRASS THEOREM** \(^{11}\)

Each bounded infinite subset of \( \mathbb{R} \) has at least one limit point.

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\(^8\)Pages 25 – 27

\(^9\)Equivalent to, but varies from, book’s def.. Book’s def. is “a compact set is a closed and bounded set”.

\(^{10}\)\( \iff \) is Thm. 1.3.7. \( \Rightarrow \) is Exercise 1.3.21.

\(^{11}\)Thm. 1.3.8