Supremum and Infimum of a Set

Summary

Set up
- \( S \subset \mathbb{R} \)
- \( \mathbb{R} \) is the set of real numbers
- \( \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \cup \{-\infty\} \) is the set of extended real numbers

Supremum of \( S = \sup S \)
also called
Least Upper Bound of \( S = \text{lub} S \)

Def.'s
- Let \( S \) be a nonempty set that is bounded above. Then \( \beta \in \mathbb{R} \) is a sup \( S \) provided
  1. \( \beta \) is an upper bound of \( S \) (i.e., \( \forall x \in S, \ x \leq \beta \))
  2. if \( b < \beta \), then \( b \) is not an upper bound of \( S \).
- Let \( S \) be a nonempty set that is not bounded above. Then \( \sup S := \infty \).
- \( \sup \emptyset := -\infty \).

Thm. Let \( S \) be a nonempty set that is bounded above. Then the sup \( S \) is the unique real number \( \beta \in \mathbb{R} \) such that
  1. \( \beta \) is an upper bound of \( S \) (i.e., \( \forall x \in S, \ x \leq \beta \))
  2. if \( b < \beta \), then \( b \) is not an upper bound of \( S \)
  2\( ' \) if \( b < \beta \), then \( \exists x_b \in S \) such that \( b < x_b \)
  2\( '' \) if \( \varepsilon > 0 \), then \( \exists x_\varepsilon \in S \) such that \( \beta - \varepsilon < x_\varepsilon \)
  2\( ''' \) if \( \varepsilon > 0 \), then \( \exists x_\varepsilon \in S \) such that \( \beta - 17\varepsilon < x_\varepsilon \).

Summary  \( \sup S \in \hat{\mathbb{R}} \).
  1. \( \sup S \in \mathbb{R} \) if and only if \( S \) is nonempty and bounded above.
  2. \( \sup S = \infty \) if and only if \( S \) is nonempty and not bounded above.
  3. \( \sup S = -\infty \) if and only if \( S = \emptyset \).

Infimum of \( S = \inf S \)
also called
Greatest Lower Bound of \( S = \text{glb} S \)

Def.'s
- Let \( S \) be a nonempty set that is bounded below. Then \( \alpha \in \mathbb{R} \) is an inf \( S \) provided
  1. \( \alpha \) is a lower bound of \( S \) (i.e., \( \forall x \in S, \ x \geq \alpha \))
  2. if \( a < \alpha \), then \( a \) is not a lower bound of \( S \).
- Let \( S \) be a nonempty set that is not bounded below. Then \( \inf S := -\infty \).
- \( \inf \emptyset := \infty \).

Thm. Let \( S \) be a nonempty set that is bounded below. Then the inf \( S \) is the unique real number \( \alpha \in \mathbb{R} \) such that
  1. \( \alpha \) is a lower bound of \( S \) (i.e., \( \forall x \in S, \ x \geq \alpha \))
  2. if \( a < \alpha \), then \( a \) is not a lower bound of \( S \)
  2\( ' \) if \( \varepsilon > 0 \), then \( \exists x_a \in S \) such that \( x_a < a \)
  2\( '' \) if \( \varepsilon > 0 \), then \( \exists x_\varepsilon \in S \) such that \( x_\varepsilon < \alpha + \varepsilon \)
  2\( ''' \) if \( \varepsilon > 0 \), then \( \exists x_\varepsilon \in S \) such that \( x_\varepsilon < \alpha + 17\varepsilon \).

Summary  \( \inf S \in \hat{\mathbb{R}} \).
  1. \( \inf S \in \mathbb{R} \) if and only if \( S \) is nonempty and bounded below.
  2. \( \inf S = -\infty \) if and only if \( S \) is nonempty and not bounded below.
  3. \( \inf S = \infty \) if and only if \( S = \emptyset \).