Do parts (a) - (i) for the following three problems.

1. \( f(x) = \cos(17x) \quad x_0 = 0 \quad J = (-\infty, \infty) = \mathbb{R}^+ \)
2. \( f(x) = (1 + x)^{-3} \quad x_0 = 0 \quad J = \left( 0, \frac{1}{2} \right) \)
3. \( f(x) = e^x \quad x_0 = 17 \quad J = (16, 19) \)

You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

Use only:
- the definition of Taylor polynomial
- the definition of Taylor series
- the theorem/error-estimate on the \( N^{th} \)-Remainder term for Taylor polynomials.

Do NOT use a known Taylor Series (i.e., do not use methods from section 10.10).

2a. Find the following. Note the first column are functions of \( x \) and the second column are numbers.

| \( f^{(0)}(x) = \) | \( (1 + \chi)^{-3} = \frac{1}{2} \) | \( f^{(0)}(x_0) = 1 \) |
| \( f^{(1)}(x) = -3 \) | \( (1 + \chi)^{-4} = -\frac{3}{2} \) | \( f^{(1)}(x_0) = -3 \) |
| \( f^{(2)}(x) = +3.4 \) | \( (1 + \chi)^{-5} = \frac{4}{2} \) | \( f^{(2)}(x_0) = +3.4 \) |
| \( f^{(3)}(x) = -3.45 \) | \( (1 + \chi)^{-6} = -\frac{5}{2} \) | \( f^{(3)}(x_0) = -3.45 \) |
| \( f^{(4)}(x) = +3.456 \) | \( (1 + \chi)^{-7} = +\frac{6}{2} \) | \( f^{(4)}(x_0) = +3.456 \) |

2b. Find the \( N^{th} \)-order Taylor polynomial of \( y = f(x) \) about \( x_0 \) in OPEN form for \( N = 0, 1, 2, 3, 4 \).

\( P_0(x) = 1 \)
\( P_1(x) = -3 \chi \)
\( P_2(x) = -3 \chi + \frac{3.4}{2} \chi^2 \)
\( P_3(x) = -3 \chi + \frac{3.4}{2} \chi^2 - \frac{3.45}{2} \chi^3 \)
\( P_4(x) = -3 \chi + \frac{3.4}{2} \chi^2 - \frac{3.45}{2} \chi^3 + \frac{3.4.5.6}{4!} \chi^4 \)

As for the order in which to do parts (c), (d), (e).
I think it is easiest to do
(1) (1c) → (1d) → (1e) but for (2) : (2e) → (2d) → (2c).
2c. Find the Taylor series of $y = f(x)$ about $x_0$ in OPEN form.

$$P_n(x) = 1 - 3x + \frac{(3)(4)}{2} x^2 - \frac{(4)(5)}{2} x^3 + \frac{(5)(6)}{2} x^4 - \frac{(6)(7)}{2} x^5 + \ldots$$

2d. Find the Taylor series of $y = f(x)$ about $x_0$ in CLOSED form.

$$P_n(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n$$

2e. Find the $n$th Taylor coefficient of $y = f(x)$ about $x_0$.

$$c_n = (-1)^n \frac{(n+1)(n+2)}{2}$$

Let make a chart to figure out $c_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{3}{1!}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{3 \cdot 4}{2!}$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{3 \cdot 4 \cdot 5}{3!}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3 \cdot 4 \cdot 5 \cdot 6}{4!}$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{5!}$</td>
</tr>
</tbody>
</table>

Also works

$$(-1)^n \frac{(n+1)(n+2)}{2}$$
2f. Find the interval of convergence \( I \) of the Taylor series of \( y = f(x) \) about \( x_0 \). Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

\[ I = (-1, 1) \]

\[
\mathbb{P}_n(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n
\]

\[
\rho(n+1)(n+3) x^{n+1} - \frac{2}{(n+1)(n+2)} x^n
\]

\[
= 1 \times \lim_{n \to \infty} \frac{n+3}{n+1} = |x| - 1 < 1
\]

Check endpoints:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (-1)^n = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} [(-1)^n (-1)^n] = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}
\]

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2}
\]

\[
\lim_{n \to \infty} \left| (\pm 1)^n \frac{(n+1)(n+2)}{2} \right| = \frac{1}{2} \lim_{n \to \infty} \frac{(n+1)(n+2)}{2} = \infty
\]

To have divergence at both ends, by the nth term test.
2g. Consider the given interval $J$ and fix an $N \in \mathbb{N}$. Find an upper bound for the maximum of $|f^{(N+1)}(x)|$ on the interval $J$. You answer can have an $N$ in it but it cannot have an $x, x_0, c$. (Note that $J$ is a subset of $I$ but Prof. G. might have picked a smaller $J$ than $I$ to make the problem easier.)

\[
\max_{c \in J} |f^{(N+1)}(c)| \leq \frac{(N+3)!}{2}
\]

\[(2a) \implies f^{(N)}(x) = (-1)^n \frac{(n+2)!}{2} (1+x)^{-(n+3)} \quad \text{So} \quad 0 < c < \frac{1}{2}
\]

\[|f^{(N+1)}(c)| \leq \left| (-1)^{N+1} \frac{(N+1)!}{2} (1+c)^{-(N+3)} \right|
\]

\[= \frac{(N+3)!}{2} \frac{1}{(1+c)^{N+4}} \leq \frac{(N+3)!}{2} \frac{1}{(1+0)^{N+4}} = \frac{(N+3)!}{2}
\]

\[0 < c < \frac{1}{2} \implies 1 < 1+c < \frac{3}{2} \implies |N+1| \leq (1+c)^{N+4} \leq \left( \frac{3}{2} \right)^{N+4}
\]

2h. Consider the given interval $J$ and fix an $N \in \mathbb{N}$. For each $x \in J$, find an upper bound for the maximum of $|R_N(x)|$.

You answer can have an $N$ and $x$ in it but it cannot have an: $x_0, c$.

\[|R_N(x)| \leq \frac{(N+2)(N+3)}{2} |x|^{N+1}
\]

\[R_N(x) \leq \max_{c \in J} |f^{(N+1)}(c)| \cdot \frac{|x|^{N+1}}{(N+1)!}
\]

\[\leq \frac{1}{2} \frac{(N+3)!}{(N+1)!} \cdot |x|^{N+1} = \frac{(N+2)(N+3)}{2} |x|^{N+1}
\]
2i. Carefully show that \( f(x) = P_n(x) \) for each \( x \) in the given interval \( J \) by showing that \( \lim_{N \to \infty} |R_N(x)| = 0 \) for each \( x \in J \).

Let \( x \in J \). So \( 0 < x < \frac{1}{2} \). So

\[
|R_N(x)| \leq \frac{(N+2)(N+3)}{2} |x|^{N+1}
\]

\[
\leq \left( \frac{N+2}{N+3} \right) \left( \frac{1}{2} \right)^{N+1}
\]

\[
= \frac{1}{2^3} \frac{(N+2)(N+3)}{2^N}
\]

\[
\xrightarrow{N \to \infty} 0
\]

\text{Way #1: L'Hôpital}

\[
\lim_{N \to \infty} \frac{(N+2)(N+3)}{2^N} = \lim_{N \to \infty} \frac{N^2 + 5N + 6}{2^N}
\]

\[
\xrightarrow{\text{L'Hôpital}} \lim_{N \to \infty} \frac{2N + 5}{2^N (\ln 2)}
\]

\[
\xrightarrow{N \to \infty} 0
\]

\text{Way #2: Helpful Intuition}

\[
0 \leq \frac{(N+2)(N+3)}{2^N} \leq \frac{(N+2)(N+3)}{N^3}
\]

\[
\xrightarrow{N \to \infty} 0
\]

So \( \lim_{N \to \infty} |P_n(x)| = 0 \)