1. Start with a sequence \( \{a_n\}_{n=1}^{\infty} \). (In above Ex., \( a_n = \frac{1}{2^n} \)). Think of \( \{a_n\}_{n} \) as an ordered list of numbers, i.e.,
\[
\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}
\]

2. Form the corresponding (formal) series \( \sum_{n=1}^{\infty} a_n \).
Think of the series \( \sum a_n \) as
\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k .
\]

3. Look at the corresponding \( n^{th} \) partial sum \( s_n \) where
\[
s_n := a_1 + a_2 + \cdots + a_n \quad {\text{def}} \quad \sum_{k=1}^{n} a_k .
\]
So we get the sequence of partial sums
\[
\{s_n\}_{n=1}^{\infty} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4 + a_5, \ldots\}.
\]

4. Beware: for a series \( \sum a_n \)
   - the \( n^{th} \) partial sum of \( \sum a_n \) is \( s_n = a_1 + a_2 + \ldots + a_n \)
   - the \( n^{th} \) term of \( \sum a_n \) is \( a_n \).
Since \( s_n = (a_1 + \ldots + a_{n-1}) + a_n = s_{n-1} + a_n \), we have that relation that \( a_n = s_n - s_{n-1} \).

5. We say that the infinite
   - 5.1) series \( \sum_{n} a_n \) converges provided the sequence of partial sums \( \{s_n\}_{n} \) converges
   - 5.2) series \( \sum_{n} a_n \) diverges to \( +\infty \) provided the sequence of partial sums \( \{s_n\}_{n} \) diverges to \( +\infty \)
   - 5.3) series \( \sum_{n} a_n \) diverges to \( -\infty \) provided the sequence of partial sums \( \{s_n\}_{n} \) diverges to \( -\infty \).
   - 5.4) series \( \sum_{n} a_n \) diverges provided the sequence of partial sums \( \{s_n\}_{n} \) diverges
We write (in the first 3 cases, i.e., in 5.1, 5.2, and 5.3) as:
\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (a_1 + a_2 + \ldots + a_n) = \lim_{n \to \infty} \sum_{k=1}^{n} a_k .
\]

6. It doesn’t matter where you start Theorem Recall: \( \sum_{k=1}^{N} a_k = (a_1 + a_2 + \ldots + a_{16}) + \sum_{k=17}^{N} a_k . \)
So \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=17}^{\infty} a_n \) do the same thing amongst the choices from 5.1–5.4.
   - \( \sum_{n=1}^{\infty} a_n \) converges (to some finite number) \( \iff \sum_{n=17}^{\infty} a_n \) converges (to some finite number).
   - (warning: each series converges but the finite number they converge to may be different).
     - \( \sum_{n=1}^{\infty} a_n \) diverges to \( \infty \) \( \iff \sum_{n=17}^{\infty} a_n \) diverges to \( \infty \).
     - \( \sum_{n=1}^{\infty} a_n \) diverges to \( -\infty \) \( \iff \sum_{n=17}^{\infty} a_n \) diverges to \( -\infty \).
     - \( \sum_{n=1}^{\infty} a_n \) diverges \( \iff \sum_{n=17}^{\infty} a_n \) diverges.

7. \( n^{th} \)-term test for divergence.
Since: If \( \sum a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).
   - get the Test: If \( \lim_{n \to \infty} a_n \neq 0 \) (which includes the possibility that \( \lim_{n \to \infty} a_n \) DNE), then \( \sum a_n \) diverges.
   - Warning: If \( \lim_{n \to \infty} a_n = 0 \), then it is possible that \( \sum a_n \) converges and it is possible that \( \sum a_n \) diverges.
   - Remark: The \( n^{th} \)-term test (for divergence) can show divergence but can NOT show convergence.

8. Let \( \sum a_n \) is a positive termed series (which just means that each term \( a_n \geq 0 \)). Then \( s_n \leq s_{n+1} \) and so the sequence \( \{s_n\}_{n} \) is \( \nearrow \); i.e., is nondecreasing and so
   - \( \text{either } \{s_n\}_{n} \text{ converges (to some finite number)} \), in which case \( \sum a_n \) converges and \( \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n \)
   - or \( \{s_n\}_{n} \) diverges to \( \infty \), in which case \( \sum a_n \) diverges to \( \infty \) and \( \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \infty \)