Definition.\(^1\) Let \(\{a_n\}_{n=1}^{\infty}\) be a sequence.

(1) The limit of the sequence \(\{a_n\}_{n=1}^{\infty}\) exists and is equal to the real number \(L\), written

\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \xrightarrow{n \to \infty} L
\]

provided:

for all positive numbers \(\epsilon\), there exists a natural number \(N\), such that

if \(n\) is a natural number bigger than \(N\), then \(L - \epsilon < a_n < L + \epsilon\).

Since \(L - \epsilon < a_n < L + \epsilon\) if and only if \(|a_n - L| < \epsilon\), condition (1) is the same as

for all positive numbers \(\epsilon\), there exists a natural number \(N\), such that

if \(n > N\), then \(|a_n - L| < \epsilon\).

Condition (1) is also the same as

for all positive numbers \(\epsilon\), there exists a natural number \(N\), such that

for all \(n > N\), \(|a_n - L| < \epsilon\).

You are welcome to use that fun/ny shorthand notation from class, which was

\(\forall\) is shorthand for \(\text{for all}\)

\(\exists\) is shorthand for \(\text{there exists}\),

to condensely rewrite the above conditions as:

\[
\forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \text{such that} \quad n > N \implies L - \epsilon < a_n < L + \epsilon.
\]

\[
\forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \text{such that} \quad n > N \implies |a_n - L| < \epsilon.
\]

\[
\forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \text{such that} \quad \forall n > N, \ |a_n - L| < \epsilon.
\]

(2) \(\{a_n\}_{n=1}^{\infty}\) converges (or is convergent) \(\iff\) you can find a real number \(L\) so that \(\lim_{n \to \infty} a_n = L\).

(3) \(\{a_n\}_{n=1}^{\infty}\) diverges (or is divergent) \(\iff\) \(\{a_n\}_{n=1}^{\infty}\) does not converge.

\(^1\)For the definition of \(\lim_{n \to \infty} a_n = L\), understand the picture/concept from class (that \(\epsilon\)-band and then go pick your \(N\) picture). You can express the definition using whichever of the 6 (equivalent) boxes that you prefer.
Example. Using the definition of limit of a sequence, show that

\[ \lim_{n \to \infty} \left( 4 - \frac{1}{17n + \ln n} \right) = 4. \]

Solution.
Fix an arbitrary \( \epsilon > 0 \).
Pick (and fix) \( N \in \mathbb{N} \) so big that

\[ N > \frac{1}{17\epsilon}. \]

Fix \( n \in \mathbb{N} \) with \( n > N \). Then

\[
\left| \left( 4 - \frac{1}{17n + \ln n} \right) - 4 \right| = \frac{1}{17n + \ln n} \leq \frac{1}{17n} \leq \frac{1}{17N} < \epsilon.
\]

We have just shown that for an arbitrary \( \epsilon > 0 \), we can pick \( N \in \mathbb{N} \) such that if \( n > N \) then \( \left| \left( 4 - \frac{1}{17n + \ln n} \right) - 4 \right| < \epsilon \). So, by definition of limit of a sequence, \( \lim_{n \to \infty} \left( 4 - \frac{1}{17n + \ln n} \right) = 4 \).

Problem. Using the definition of limit of a sequence, show that

\[ \lim_{n \to \infty} \frac{3n^2 - \sin n}{16n^2} = \frac{3}{16}. \]

Note that solutions may vary.

Solution.
Fix an arbitrary \( \epsilon > 0 \).
Pick (and fix) \( N \in \mathbb{N} \) so big that

\[ N > \frac{1}{4\sqrt{\epsilon}}. \]

Fix \( n \in \mathbb{N} \) with \( n > N \). Then (using, as in class, \( \Delta \) to mean that a step follows from algebra)

\[
\left| \left( \frac{3n^2 - \sin n}{16n^2} \right) - \frac{3}{16} \right| = \left| \frac{\frac{3n^2}{16n^2} - \frac{\sin n}{16n^2} - \frac{3}{16} \right| \leq \left| \frac{\sin n}{16n^2} \right| \leq \frac{1}{16n^2} \leq \frac{1}{16N^2} \leq \frac{1}{16\left( \frac{1}{4\sqrt{\epsilon}} \right)^2} = \frac{1}{16\left( \frac{1}{16\epsilon} \right)} = \epsilon.
\]

We have just shown that for an arbitrary \( \epsilon > 0 \), we can pick \( N \in \mathbb{N} \) such that if \( n > N \) then \( \left| \left( \frac{3n^2 - \sin n}{16n^2} \right) - \frac{3}{16} \right| < \epsilon \). So, by definition of limit of a sequence, \( \lim_{n \to \infty} \frac{3n^2 - \sin n}{16n^2} = \frac{3}{16} \).