Goal (for the next several sections):

Given a specific series $\sum a_n$, determine the behavior $\sum a_n$.

We know that the series $\sum a_n$ does one, and precisely one, of the below four possibilities.

1. The series $\sum a_n$ converges (to a finite number). We will not worry about finding its sum.
2. The series $\sum a_n$ diverges to $+\infty$.
3. The series $\sum a_n$ diverges to $-\infty$.
4. The series $\sum a_n$ diverges but does not diverge to $\pm\infty$.

Recall that a series $\sum a_n$ does the same thing as the limit of the sequence $\{s_n\}$ of its partial sums.

**Key Idea Behind Positive-Termed Series**

Definition: $\sum a_n$ is a positive-term series if $a_n \geq 0$ for each $n$.

Explore: Let $\sum a_n$ be a positive-term series.

(i) Consider its **sequence** of partial sums $\{s_n\}_{n \in \mathbb{N}}$ where $s_n = a_1 + a_2 + \ldots + a_n$.

(ii) Recall that $\sum_{n=1}^{\infty} a_n \overset{\text{def}}{=} \lim_{n \to \infty} s_n$.

(iii) $[a_n \geq 0 \text{ for each } n \in \mathbb{N}] \implies [s_n \leq s_{n+1} \text{ for each } N \in \mathbb{N}]$.

(iv) Thus $\{s_n\}_{n \in \mathbb{N}}$ is an increasing sequence.

(v) So we can apply the monotone sequence theorem\(^1\) to $\{s_n\}_{n \in \mathbb{N}}$.

So either:

- $\{s_n\}_{n \in \mathbb{N}}$ is bounded above,\(^2\) in which case, by the monotonic sequence theorem, $\lim_{n \to \infty} s_n$ exists (as a **finite** real number) and thus by (ii) above, $\sum a_n$ converges (to a **finite** real number)

or

- $\{s_n\}_{n \in \mathbb{N}}$ is not bounded above, in which case, by the monotonic sequence theorem, $\lim_{n \to \infty} s_n = \infty$ and thus by (ii) above, $\sum a_n$ diverges (to $\infty$).

**Positive-Termed Series Criteria**

Let $\sum a_n$ be a **positive-term** series. Set 

$$ s_n := \sum_{k=1}^{n} a_k, $$

Then either

- there exists a bound $B$ so that for each $n$ we have $s_n \leq B$, in which case, the series $\sum a_n$ converges (to a **finite** real number)

or

- $\lim_{n \to \infty} s_n = \infty$, in which case, the series $\sum a_n$ diverges (to $\infty$).

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\(^1\)$§10.1 Theorem 6.

\(^2\)Recall: a sequence $\{s_n\}_{n \in \mathbb{N}}$ is bounded above means that there is a $B \in \mathbb{R}$ such that $s_n \leq B$ for each $n \in \mathbb{N}$. 

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Prof. Girardi, 16.02.29

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Infinite Series