Throughout this handout, we are dealing with a function \( f: [a, b] \rightarrow \mathbb{R} \).

In recent sections, often our goal was to find the antiderivative \( F \) of \( f \), i.e. to find a function \( F: [a, b] \rightarrow \mathbb{R} \) such that
\[
\int f(x) \, dx = F(x) + C.
\]

Now goal is to find the number \( \int_a^b f(x) \, dx \).

If we can find the antiderivative \( F \), then we know (FTC) that
\[
\int_a^b f(x) \, dx = \int_a^b F'(x) \, dx = F(b) - F(a) .
\]

The trouble is that sometimes \( F \) is hard to find . . . so we cope . . . and numerically approximate \( \int_a^b f(x) \, dx \), i.e., we look for some number \( L \in \mathbb{R} \) so that
\[
\int f(x) \, dx \approx L .
\]

We learned (§5.1,5.2) how to numerically approximate \( \int_a^b f(x) \, dx \) using Riemann Sums/Rectangles. Now we are going to learn the Trapezoidal (Trap.) Rule, which uses trapezoids instead of rectangles.

To use the Trap. Rule, start off just as you would when forming Riemann Rectangles.

- Divide \([a, b]\) into \( n \) subintervals \([x_{i-1}, x_i]\), each of length
  \[
  \Delta x = \frac{b - a}{n}
  \]

- So we get
  \[
  x_i = a + i (\Delta x)
  \]

for \( i = 0, 1, 2, \ldots, n \).

Trapezoidal Rule approximation \( T_n \) of \( \int_a^b f(x) \, dx \) with \( n \)-steps is
\[
T_n = \frac{\Delta x}{2} \left[ f(x_0) + \left( 2 \sum_{i=1}^{n-1} f(x_i) \right) + f(x_n) \right].
\]

Question. \( \int_a^b f(x) \, dx \approx T_n \) but how good is this approximation \( \approx \)?

Answer. The Trapezoidal Rule Error Theorem tells us!

Trapezoidal Rule Error Theorem.

Let \( f'' \) be continuous on \([a, b]\).

Set \( M_0 := \max_{a \leq x \leq b} |f''(x)| \) and let \( M_0 \leq M \).

Then
\[
\left| T_n - \int_a^b f(x) \, dx \right| \leq \frac{M (b - a)^3}{12n^2}.
\]
Let
\[ f(x) = \sqrt{x} \quad \text{and} \quad [a, b] = [4, 5] \quad \text{and} \quad n = 3. \]

1. Find the Trapezoidal Rule approximation \( T_n \) of \( \int_a^b f(x) \, dx \) with \( n \) steps.

ANSWER: \[ T_n = \]

2. Find a good upper bound for \( \max_{a \leq x \leq b} |f''(x)| \).

ANSWER: \[ \max_{a \leq x \leq b} |f''(x)| \leq \]

3. The Trapezoidal Rule Error Theorem gives that \( \left| T_n - \int_a^b f(x) \, dx \right| \leq \)

Your answer should be a number but you do not have to perform grade school level arithmetic.

4. Find the smallest integer \( n \) so that the Trapezoidal Rule Error Theorem guarantees that \( \left| T_n - \int_a^b f(x) \, dx \right| \leq 10^{-4} \).

ANSWER: \( n = \)

Recall: \( 1 \times 10^{-4} = 0.0001 \)