Determine the behavior of the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \). We will show that
\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} 
\text{convergent} & \text{if } p > 1 \\
\text{divergent to } +\infty & \text{if } p \leq 1 .
\end{cases}
\]
When \( p = 1 \), the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is also called the harmonic series.

Note that the \( p \)-series looks like
\[
\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \\
= 1 + \left( \frac{1}{2} \right)^p + \left( \frac{1}{3} \right)^p + \left( \frac{1}{4} \right)^p + \ldots
\]
Do not confuse a \( p \)-series \( \sum_{n=1} \frac{1}{n^p} \) with a geometric series \( \sum_{n=1} r^n \).

Recall that in the section on Improper Integrals, we showed that
\[
\int_{n=1}^{\infty} \frac{1}{x^p} \, dx \quad \text{is} \quad \begin{cases} 
\text{convergent to } \frac{1}{p-1} & \text{if } p > 1 \\
\text{divergent to } +\infty & \text{if } p \leq 1 .
\end{cases}
\]

Let’s say we are given a series \( \sum a_n \) and we can find a function \( f: \left[ \frac{1}{n}, \infty \right) \to \mathbb{R} \) satisfying
(1) \( a_n = f(n) \) for each \( n \in \mathbb{N} \) with \( n \geq \frac{1}{n} \) (this is usually accomplished by design)
(2) \( y = f(x) \) is positive on \( \left[ \frac{1}{n}, \infty \right) \) (so \( \sum a_n \) needs to be a positive term series)
(3) \( y = f(x) \) is continuous on \( \left[ \frac{1}{n}, \infty \right) \)
(4) \( y = f(x) \) is decreasing on \( \left[ \frac{1}{n}, \infty \right) \) (can confirm this by showing \( f'(x) \leq 0 \)).

Then the series \( \sum_{n=1}^{\infty} a_n \) and the improper integral \( \int_{x=1/n}^{\infty} f(x) \, dx \) either:
(a) both converge (to finite numbers, although most likely different numbers)
(b) both diverge (to \( \infty \)).
Let’s say we are given a series $\sum a_n$ and find a function $f : [1, \infty) \rightarrow \mathbb{R}$ satisfying (1)–(4). Then the sequence $\{\sum_{k=1}^{n} a_k\}_{n \in \mathbb{N}}$ and the sequence $\{\int_{1}^{n} f(x) \, dx\}_{n \in \mathbb{N}}$ are both increasing sequences and so each sequence has the choice of either [converging to some finite number] or [diverging to $\infty$].

Next compare the terms of these two sequences:

\[
\sum_{k=1}^{n} a_k \leq \int_{1}^{n} f(x) \, dx \leq \sum_{k=1}^{n-1} a_k .
\]

Now take the limit as $n \rightarrow \infty$ to see that

\[
\sum_{k=2}^{\infty} a_k \leq \int_{1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} a_k . \quad (\star_1)
\]

The integral test now follows from $(\star_1)$.

What if we changed our interval $[1, \infty)$ to $[17, \infty)$? Then $(\star_1)$ would become

\[
\sum_{k=17+1}^{\infty} a_k \leq \int_{17}^{\infty} f(x) \, dx \leq \sum_{k=17}^{\infty} a_k . \quad (\star_{17})
\]

**Observation 1.** The statement of the Integral Test remains true if we replace each $1$ with $17$, or any other integer. This is useful if, e.g., you can get (1)–(3) to hold but only have $y = f(x)$ decreasing on $[17, \infty)$.

**Observation 2.** The Integral Test Remainder Estimate.

Let’s say that we have shown that $\sum a_n$ converges by using the integral test with the function $y = f(x)$, which satisfies that above conditions (1) - (4). Then we can approximate the infinite sum $S := \sum_{n=1}^{\infty} a_n$ by the computable finite sum $s_n := \sum_{k=1}^{n} a_k$. Indeed, define $S$, $s_n$, and $R_n$ by

\[
S := \sum_{n=1}^{\infty} a_n \quad \text{and} \quad s_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad S = s_n + R_n .
\]

Then $S \approx s_n$ within an error of $|R_n|$ and we can bound $R_n$ by

\[
0 \leq \int_{x=n+1}^{\infty} f(x) \, dx \leq R_n \text{ note } \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k \text{ note } \sum_{k=n+1}^{\infty} a_k \leq \int_{x=n}^{\infty} f(x) \, dx .
\]