INSTRUCTIONS

• The MARK BOX above indicates the problems along with their points. Check that your copy of the exam has all of the problems.

• This exam comes in two parts.
  (1) HAND IN PART. Hand in only this part, which includes a table for indicating your solutions to the multiple choice problems.
  (2) STATEMENT OF MULTIPLE CHOICE PROBLEMS part. Do not hand in this part. You can take this part home to learn from and to check your answers once the solutions are posted.

• Upon request, you will be given as much (blank) scratch paper as you need.

• During the exam, the use of unauthorized materials is prohibited. Unauthorized materials include: electronic devices, books, and personal notes. Unauthorized materials (including cell phones) must be in a secured (e.g. zipped up, snapped closed) bag placed completely under your desk or, if you did not bring such a bag, given to Prof. Girardi to hold for you during the exam (and they will be returned when you leave the exam). This means no electronic devices (such as cell phones) allowed in your pockets. Please, if I forget, remind me to pull up a clock on the projector screen.

• During this exam, do not leave your seat unless you have permission. If you have a question, raise your hand. When you finish: put your pencil down and raise your hand.

• This exam covers (from Calculus by Stewart, 6th ed., ET):

  §7.1–7.5, 7.8, 11.1 – 11.11, 6.1 – 6.3, 10.3–10.4 .

Honor Code Statement
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina’s Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

I understand that if it is determined that I used any unauthorized assistance or otherwise violated the University’s Honor Code then I will receive a failing grade for this course and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary actions.

Furthermore, I have not only read but will also follow the above Instructions.

Signature:
Indicate (by circling) directly in the table below your solution to each problem.

- You may choose up to 2 answers for each problem. The scoring is as follows. For a problem with precisely one answer marked and the answer is correct, 4 points. For a problem with precisely two answers marked, one of which is correct, 1 point. All other cases, 0 points.
- Fill in the “number of solutions circled” column.

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• Hint for a typical (i.e. not improper) definite integral problems \( \int_a^b f(x) \, dx \). First do the indefinite integral, say you get \( \int f(x) \, dx = F(x) + C \). To check if you did this part correctly, you can use the Fundamental Theorem of Calculus (i.e. \( F'(x) \) should be \( f(x) \)). Once you are confident that your indefinite integral is correct, use the indefinite integral to find the definite integral.

• **Hint:** \( \ln b - \ln a = \ln \left( \frac{b}{a} \right) \) and \( \ln(a^r) = r \ln a \) if \( a,b > 0 \) and \( r \in \mathbb{R} \).

1. \[ \int_{x=0}^{x=1} \frac{1}{x^2+1} \, dx = \left. \arctan x \right|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} . \]

2. \[ \int_{x=0}^{x=1} \frac{x}{x^2+1} \, dx = \frac{1}{2} \int_{x=0}^{x=1} \frac{2x \, dx}{x^2+1} = \frac{1}{2} \ln |x^2+1| \bigg|_{x=0}^{x=1} = \frac{1}{2} \ln (2) - \frac{1}{2} \ln (1) = \frac{1}{2} \ln (2) - 0 = \ln \sqrt{2} . \]

3. Evaluate the integral \( \int_{x=0}^{x=e} \ln x \, dx \).

Note \( \lim_{a \to 0^+} \ln a = -\infty \) and so we have an improper integral: \( \int_{x=0}^{x=e} \ln x \, dx = \lim_{a \to 0^+} \int_{x=a}^{x=e} \ln x \, dx \).

**Example 2** Evaluate \( \int \ln x \, dx \).

**Solution** Here we don’t have much choice for \( u \) and \( dv \). Let

\[ u = \ln x \quad dv = dx \]

Then

\[ du = \frac{1}{x} \, dx \quad v = x \]

Integrating by parts, we get

\[ \int \ln x \, dx = x \ln x - \int x \frac{dx}{x} \]

\[ = x \ln x - \int dx \]

\[ = x \ln x - x + C \]

Integration by parts is effective in this example because the derivative of the function \( f(x) = \ln x \) is simpler than \( f \).

If \( a > 0 \), then

\[ \int_{x=a}^{x=e} \ln x \, dx = (x \ln x - x) \bigg|_{x=a}^{x=e} = (e \ln e - e) - (a \ln a - a) = -a \ln a + a = a (1 - \ln a) . \]

So

\[ \int_{x=0}^{x=e} \ln x \, dx = \lim_{a \to 0^+} \int_{x=a}^{x=e} \ln x \, dx = \lim_{a \to 0^+} a (1 - \ln a) \overset{(0)(\infty)}{=} \lim_{a \to 0^+} \frac{1 - \ln a}{a^{-1}} \overset{L'H}{=} \lim_{a \to 0^+} -1/a = \lim_{a \to 0^+} a = 0 . \]
4. \( \int_{x=0}^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx = \left( \frac{1}{5} - \frac{1}{7} \right) - (0 - 0) = \frac{2}{35}. \)

Example 4 Evaluate \( \int \cos^3 x \sin^4 x \, dx. \)

Solution We proceed as follows:

\[
\int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \sin^4 x \cos x \, dx \\
= \int (1 - \sin^2 x) \sin^4 x \cos x \, dx.
\]

If we let \( u = \sin x \), then \( du = \cos x \, dx \) and the integral may be written

\[
\int \cos^3 x \sin^4 x \, dx = \int (1 - u^2)u^4 \, du \\
= \int (u^4 - u^6) \, du \\
= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C \\
= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C.
\]

5. \( \int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} \, dx = \ln \left| \frac{x^3(x-1)}{x+3} \right| \bigg|_{\frac{3}{2}}^{3} = \ln \left| \frac{3^3}{\frac{3}{2}^3} \cdot \frac{5}{2^{3/2}} \right| = \ln \left( \frac{325}{3} \right) = \ln \frac{45}{5}. \)

Example 1 Evaluate \( \int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} \, dx. \)

Solution The denominator of the integrand has the factored form \( x(x + 3)(x - 1) \). Each of the linear factors is handled under Rule 1, with \( m = 1 \). Thus, for the factor \( x \) there corresponds a partial fraction of the form \( A/x \). Similarly, for the factors \( x + 3 \) and \( x - 1 \) there correspond partial fractions \( B/(x + 3) \) and \( C/(x - 1) \), respectively. The decomposition (10.3) then has the form

\[
\frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x - 1}.
\]

Multiplying by the lowest common denominator gives us

(10.4) \( 4x^2 + 13x - 9 = A(x + 3)(x - 1) + Bx(x - 1) + Cx(x + 3). \)
Multiplying by the lowest common denominator gives us

$$4x^2 + 13x - 9 = A(x + 3)(x - 1) + Bx(x - 1) + Cx(x + 3).$$

In a case such as this, in which the factors are all linear and nonrepeated, the values for $A$, $B$, and $C$ can be found by substituting values for $x$ which make the various factors zero. If we let $x = 0$ in (10.4), then

$$-9 = -3A \quad \text{or} \quad A = 3.$$  

Letting $x = 1$ in (10.4) gives us

$$8 = 4C \quad \text{or} \quad C = 2.$$  

Finally, if $x = -3$, then

$$-12 = 12B \quad \text{or} \quad B = -1.$$  

The partial fraction decomposition is, therefore,

$$\frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)} = \frac{3}{x} + \frac{-1}{x + 3} + \frac{2}{x - 1}.$$  

Integrating,

$$\int \frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)} \, dx = \int \frac{3}{x} \, dx + \int \frac{-1}{x + 3} \, dx + \int \frac{2}{x - 1} \, dx$$

$$= 3 \ln |x| - \ln |x + 3| + 2 \ln |x - 1| + D$$

$$= \ln |x^3| - \ln |x + 3| + \ln |x - 1|^2 + D$$

$$= \ln \left| \frac{x^3(x - 1)^2}{x + 3} \right| + D.$$  

Another technique for finding $A$, $B$, and $C$ is to compare coefficients of $x$. If the right-hand side of (10.4) is expanded and like powers of $x$ are collected, then

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A.$$  

We now use the fact that if two polynomials are equal, then coefficients of like powers are the same. Thus

$$\begin{align*}
A + B + C &= 4 \\
2A - B + 3C &= 13 \\
-3A &= -9.
\end{align*}$$

It is left to the reader to show that the solution of this system of equations is $A = 3$, $B = -1$, and $C = 2$.

6. $\int_0^1 \frac{1}{\sqrt{x^2 + 8x + 25}} \, dx = \ln \left| \sqrt{x^2 + 8x + 25} + x + 4 \right|_0^1 = \ln \left| \sqrt{34} + 5 \right| - \ln \left| \sqrt{25} + 4 \right| = \ln \frac{\sqrt{34} + 5}{9}$

Hint: $x^2 + 8x + 25 = (x + 4)^2 + 9$.  

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Example 3 Evaluate \( \int \frac{1}{\sqrt{x^2 + 8x + 25}} \, dx \).

Solution We complete the square for the quadratic expression as follows:

\[
x^2 + 8x + 25 = (x^2 + 8x + 16) + 9
= (x + 4)^2 + 9.
\]

Hence

\[
\int \frac{1}{\sqrt{x^2 + 8x + 25}} \, dx = \int \frac{1}{\sqrt{(x + 4)^2 + 9}} \, dx.
\]

If we next make the trigonometric substitution

\[
x + 4 = 3 \tan \theta
\]

then

\[
dx = 3 \sec^2 \theta \, d\theta
\]

\[
\sqrt{(x + 4)^2 + 9} = \sqrt{9 \tan^2 \theta + 9} = 3 \tan \theta + 1 = 3 \sec \theta
\]

and hence

\[
\int \frac{1}{\sqrt{x^2 + 8x + 25}} \, dx = \int \frac{1}{3 \sec \theta} \cdot 3 \sec^2 \theta \, d\theta
= \int \sec \theta \, d\theta
\]

\[
= \ln |\sec \theta + \tan \theta| + C.
\]

In order to return to the variable \( x \) we use the triangle in Figure 10.4. This gives us

\[
\int \frac{1}{\sqrt{x^2 + 8x + 25}} \, dx = \ln \left| \frac{\sqrt{x^2 + 8x + 25}}{3} + \frac{x + 4}{3} \right| + C
\]

\[
= \ln \left| \sqrt{x^2 + 8x + 25 + x + 4} \right| + D
\]

where \( D = C - \ln 3 \).
7. \[ \int_{x=0}^{\pi/2} \cos^4 x \, dx = \left( \frac{3}{8} x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) \right) \bigg|_0^{\pi/2} = \left( \frac{3\pi}{16} - 0 - 0 \right) - (0 - 0 + 0) = \frac{3\pi}{16}. \]

Example 4  Two successive applications of the half-angle formula for the cosine give
\[
\cos^4 x = (\cos^2 x)^2 = \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{4}(1 + 2 \cos 2x + \cos^2 2x)
= \frac{1}{4}(1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x))
= \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x,
\]
so
\[
\int \cos^4 x \, dx = \frac{1}{4}x + \frac{1}{4} \sin 2x + \frac{1}{16} \sin 4x + C.
\]

8. \[ \int_{x=0}^{\pi/2} e^x \cos x \, dx = \frac{e^x (\sin x + \cos x)}{2} \bigg|_0^{\pi/2} = \frac{e^{3\pi/2}(-1)}{2} - \frac{e^0(1)}{2} = \frac{-1 - e^{3\pi/2}}{2}. \]

Example 5  Find \( \int e^x \cos x \, dx \).

Solution  Let
\[
\begin{align*}
u &= e^x & dv &= \cos x \, dx \\
du &= e^x \, dx & v &= \sin x.
\end{align*}
\]
Integrating by parts,
\[
(a) \quad \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.
\]
We next apply integration by parts to the integral on the right side of equation (a). Letting
\[
\begin{align*}
u &= e^x & dv &= \sin x \, dx \\
du &= e^x \, dx & v &= -\cos x,
\end{align*}
\]
and integrating by parts leads to
\[
(b) \quad \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.
\]
If we now use equation (b) to substitute on the right side of equation (a) we obtain
\[
\int e^x \cos x \, dx = e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x \, dx \right]
\]
or
\[
\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.
\]
Adding \( \int e^x \cos x \, dx \) to both sides gives us
\[
2 \int e^x \cos x \, dx = e^x (\sin x + \cos x).
\]
Finally, dividing both sides by 2 and adding the constant of integration, we have
\[
\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C.
\]
9. \( \int_{x=-1}^{x=1} \frac{1}{x^{3/4}} \, dx = \lim_{a \to 0^-} \int_{x=-1}^{x=a} \frac{1}{x^{3/4}} \, dx + \lim_{b \to 0^+} \int_{x=b}^{x=1} \frac{1}{x^{3/4}} \, dx \) \( \Rightarrow \) \( 3 + 3 = 6 \).

Note that \( \int_{x=0}^{x=1} x^{-3/4} \, dx = \lim_{b \to 0^+} \int_{x=0}^{x=b} x^{-3/4} \, dx = \lim_{b \to 0^+} \left( \frac{3}{4} b^{1/4} \right) = 3 \).

Similarly (also by symmetry) \( \lim_{a \to 0^-} \int_{x=-1}^{x=a} x^{-3/4} \, dx = 3 \).

10. \( \int_{x=-1}^{x=1} \frac{1}{x^2} \, dx \) does not exist but also does not diverge to infinity.

11. \( \frac{\sqrt{25n^8 + 5n^7 - n^2 + 1}}{3n^4 + 5n^3 - n - 2} \), \( \frac{1}{n^k} \) \( \Rightarrow \) \( \frac{\sqrt{25 + 5n^{-1} - n^{-6} + n^{-8}}}{3 + 5n^{-2} - n^{-3} - 2n^{-4}} \) \( \to \infty \) \( \frac{5}{3} \).

12. For what value \( r \in \mathbb{R} \) does \( \sum_{n=2}^{\infty} r^n = \frac{1}{4} \) ? Answer: when \( r = \frac{-1 + \sqrt{17}}{8} \).

Let \( s_N = \sum_{n=2}^{N} r^n \). Then

\[
s_N = r^2 + r^3 + \ldots + r^N
\]

\[
s_N = r^3 + r^4 + \ldots + r^{N+1}.
\]

So if \( \left| r \right| < 1 \), then

\[
s_N = \frac{r^2 - r^{N+1}}{1 - r} \quad \lim_{N \to \infty} \quad \frac{r^2}{1 - r}.
\]

So we want \( \frac{r^2}{1 - r} = \frac{1}{1} \). And

\[
\frac{r^2}{1 - r} = \frac{1}{4} \iff 4r^2 = 1 - r \iff 4r^2 + r - 1 = 0 \iff r = -1 \pm \frac{\sqrt{1 - (4)(1)(-1)}}{2} \iff r = -1 \pm \frac{\sqrt{17}}{8}.
\]

Note \( 4 = \sqrt{16} < \sqrt{17} < \sqrt{25} = 5 \) and so

\[
0 < \frac{3}{8} < -\frac{1 + \sqrt{17}}{8} < \frac{4}{8} < 1 \quad \text{and} \quad -1 < -\frac{6}{8} < -\frac{1 - \sqrt{17}}{8} < \frac{-5}{8} < 0
\]

and so \( \left| \frac{-1 + \sqrt{17}}{8} \right| < 1 \).
13. The formal series \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{(n+2)(n+7)}} \) converges conditionally as can be shown by using the LCT with \( b_n = \frac{1}{n} \) as well as the AST.
14. $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$ converges, as can be shown by the integral test.

The function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_{2}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \left[ \frac{-1}{\ln x} \right]_{2}^{t}$$

(by substitution with $u = \ln x$)

$$= -\lim_{t \to \infty} \left( \frac{1}{\ln t} - \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

so the series $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$ converges.

15. $\sum_{n=2}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n$ converges by the Root Test.

**Example 6** Test the convergence of the series $\sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n$.

**Solution**

Let $a_n = \left( \frac{2n+3}{3n+2} \right)^n$ and $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!}{n!} \cdot \frac{(cn+c)!}{(cn)!} \right| = \frac{(n+1)^6}{(cn+c)!} \cdot \frac{(cn)!}{(cn+1)(cn+2)\cdots(cn+c)}$$

$$= \frac{n^6}{c^n (n^C)} + \text{(a poly. of degree at most (c-1))}$$

If $c < 6$, then $\rho = \infty$. If $c > 6$, then $\rho = 0$. If $c = 6$, then $\rho = \frac{1}{6^6} < 1$. Now apply the ratio test.

16. Let $c$ be a natural number (i.e., $c \in \{1, 2, 3, 4, \ldots \}$).

The series $\sum_{n=1}^{\infty} \frac{(n!)^6}{(cn)!^6}$ diverges when $c < 6$ and converges when $c \geq 6$.

Let $a_n = \frac{(n!)^6}{(cn)!^6}$ and $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!^6}{n!^6} \cdot \frac{(cn+c)!}{(cn)!} \right| = \frac{(n+1)^6}{(cn+c)!} \cdot \frac{(cn)!}{(cn+1)(cn+2)\cdots(cn+c)}$$

$$= \frac{n^6}{c^n (n^C)} + \text{(a poly. of degree at most (c-1))}$$

If $c < 6$, then $\rho = \infty$. If $c > 6$, then $\rho = 0$. If $c = 6$, then $\rho = \frac{1}{6^6} < 1$. Now apply the ratio test.

17. What is the LARGEST set for which the formal power series $\sum_{n=17}^{\infty} \frac{x^n}{n!}$ is convergent (either absolutely or conditionally, so, in other words, its interval of convergence)? Answer: $(-\infty, +\infty)$
18. We know that the geometric series
\[ \sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}. \]

So if \(|2x^3| < 1\), i.e. if \(x \in (-2^{-1/3}, 2^{-1/3})\), then \(\frac{1}{1-2x^3} = \sum_{n=0}^{\infty} (2x^3)^n = \sum_{n=0}^{\infty} 2^n x^{3n}\).

19. We know that the geometric series
\[ \sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}. \]

So \(\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty} (2(x-1))^n = \sum_{n=0}^{\infty} 2^n (x-1)^n\)

This expansion is valid when \(|2(x-1)| < 1\), i.e., when \(0.5 < x < 1.5\).

20. The 3rd order Taylor polynomial for \(f(x) = \frac{1}{x}\) about the center \(x_0 = 2\) is
\[ P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 \]

21. Answer: \(|R_5(x)| \leq \frac{e^{(3^5)}}{6!}\)

For each \(x \in (-1, 3)\), there exists \(c \in (-1, 3)\) so that
\[ |R_5(x)| = \left| \frac{f^{(6)}(c)}{6!} (x-0)^6 \right| = \frac{1}{6!} e^{-c} |x|^6 \leq \frac{1}{6!} e^{-(1)} 3^6 \]

22. Answer: \(x^2 + (y-1)^2 = 1\)

\[ r = 2 \sin \theta \iff r^2 = 2r \sin \theta \]
\[ \iff x^2 + y^2 = 2y \]
\[ \iff x^2 + y^2 - 2y + 1 = 1 \]
\[ \iff x^2 + (y-1)^2 = 1, \text{ circle with radius } 1 \text{, center } (0, 1) \]

\[ \text{ ok, but can you do better? what is it? } \]
23. The area enclosed by $r = 5 - 5 \sin \theta$ is $\frac{1}{2} \int_0^{2\pi} [5 - 5 \sin \theta]^2 \, d\theta$

$\frac{1}{4}$ (period) = $\frac{\frac{2\pi}{4}}{1} = \frac{\pi}{2}$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$5 \sin \theta$</th>
<th>$r = 5 - 5 \sin \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \rightarrow \frac{\pi}{2}$</td>
<td>$0 \rightarrow 5$</td>
<td>$5 \rightarrow 0$</td>
</tr>
<tr>
<td>$\frac{\pi}{2} \rightarrow \pi$</td>
<td>$5 \rightarrow 0$</td>
<td>$0 \rightarrow 5$</td>
</tr>
<tr>
<td>$\pi \rightarrow \frac{3\pi}{2}$</td>
<td>$0 \rightarrow -5$</td>
<td>$5 \rightarrow 10$</td>
</tr>
<tr>
<td>$\frac{3\pi}{2} \rightarrow 2\pi$</td>
<td>$-5 \rightarrow 0$</td>
<td>$10 \rightarrow 5$</td>
</tr>
</tbody>
</table>

Now consider a function $r = f(\theta)$ which determines a curve in the plane where

(1) $f : [\alpha, \beta] \rightarrow [0, \infty]$ 
(2) $f$ is continuous on $[\alpha, \beta]$ 
(3) $\beta - \alpha \leq 2\pi$.

Then the area bounded by polar curves $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta.$$
24. The volume of $V$ is $2\pi \int_{x=0}^{x=1} (y) (\sqrt{y} - y) \, dy$

25. Of course, my answer is 17. What is yours?