Integration by Parts

Idea: \( \int u \, dv = uv - \int v \, du \), where \( u \) and \( v \) are functions (most likely of \( x \)) and where \( du \) and \( dv \) denote the derivatives of \( u \) and \( v \) respectively (with respect to \( x \)).

Ex: \( \int \ln x \, dx \)

Let \( u = \ln x \) and \( dv = dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = x \). So

\[
\int \ln x \, dx = \int u \, dv = uv - \int v \, du = (\ln x) \cdot x - \int x \left( \frac{1}{x} \, dx \right)
\]

\[
= x \ln x - \int dx = x \ln x - x + C.
\]

Keep in mind, \( u \)dv must cover (multiplicatively) everything in the integral. Don’t leave any bits out! Also, you cannot split portions of the function located under an inner function, so if something is under a square root/ all being squired/ in a sine/ etc, it needs to stay together!
Ex: \( \int xe^x \, dx \)

\[ u = x \quad dv = e^x \, dx \quad \text{gives} \]
\[ du = dx \quad v = e^x \]

\[ \int xe^x \, dx = uv - \int v \, du = xe^x - \int e^x \, dx \]
\[ = xe^x - e^x + C. \]

Ex: Integration by parts may be applied multiple times, as in this example.

Take \( \int e^x \sin x \, dx \)

Let \( u = \sin x \quad dv = e^x \, dx \)
\[ du = \cos x \, dx \quad v = e^x \]

So \( \int e^x \sin x \, dx = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx \).

Consider \( \int e^x \cos x \, dx \), and let \( u = \cos x \quad dv = e^x \, dx \)
\[ du = -\sin x \, dx \quad v = e^x \]

Then \( \int e^x \cos x \, dx = uv - \int v \, du = e^x \cos x - \int e^x (-\sin x) \, dx \)
\[ = e^x \cos x + \int e^x \sin x \, dx. \]

Hence, \( \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - (e^x \cos x + \int e^x \sin x \, dx) \)
\[ = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx. \]

So, \( 2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x \), and \( \int e^x \sin x \, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + C. \)
Trig Integrals

The number one thing to know for trig integrals are all your trig identities. Once you have those, solving an integral usually amounts to making an appropriate u-substitution or by applying integration by parts.

Important identities (especially popular ones are *d):

\[ \sin^3 x = \frac{1 - \cos 2x}{2} \]

\[ \sin^3 x + \cos^3 x = 1 \]

\[ \tan^3 x + 1 = \sec^3 x \]

\[ \cot^3 x = \csc^3 x \]

\[ \sin(\alpha) \cos(\beta) = \frac{1}{2} \left( \sin(\alpha - \beta) + \sin(\alpha + \beta) \right) \]

\[ \cos(\alpha) \cos(\beta) = \frac{1}{2} \left( \cos(\alpha - \beta) + \cos(\alpha + \beta) \right) \]

\[ \sin(\alpha) \sin(\beta) = \frac{1}{2} \left( \cos(\alpha - \beta) - \cos(\alpha + \beta) \right) \]

\[ \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \]

\[ \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \]

Ex: \[ \int \cos^3 x \, dx \]

\[ = \int (\cos^2 x \cos x) \, dx \]

\[ = \int \left( \frac{1 + \cos 2x}{2} \right) \cos x \, dx \]

\[ = \frac{1}{4} \int (1 + \cos 2x + \cos^2 x) \, dx \]

\[ = \frac{1}{4} \int 1 + \cos 2x + \frac{1 + \cos 4x}{2} \, dx \]

\[ = \frac{1}{8} \int 2 + 4 \cos 2x + 1 + \cos 4x \, dx \]

\[ = \frac{1}{8} \left( 3x + 2 \sin 2x + \frac{1}{4} \sin 4x \right) + C \]
Trig Substitution

Idea: replace portions of a difficult integral with trig things from an appropriately defined triangle.

Ex: \( \int \frac{1}{x^3 + 1} \, dx \)

\[ \sin \theta = x \quad \Rightarrow \quad dx = \frac{\sec^2 \theta}{\theta} \, d\theta \]
\[ \cos \theta = \frac{1}{\sqrt{x^2 + 1}} \quad \Rightarrow \quad \cos^3 \theta = \frac{1}{x^2 + 1} \]

Hence, \( \int \frac{1}{x^3 + 1} \, dx = \int \cos^3 \theta (\sec^2 \theta) \, d\theta = \int d\theta = \theta + C \)

However, \( \tan \theta = x \quad \Rightarrow \quad \theta = \arctan x \). Therefore,

\[ \int \frac{1}{x^2 + 1} \, dx = \theta + C = \arctan x + C \]
Ex: \[ \int \frac{1}{x \sqrt{1 + x^2}} \, dx \]

\[ \tan \theta = x \Rightarrow \sec^2 \theta \, d\theta = \frac{d\theta}{x} \]
\[ \sec \theta = Y \]
\[ \cos \theta = \sqrt{1 + x^2} \]

Hence, \[ \int \frac{1}{x \sqrt{1 + x^2}} \, dx = \int \frac{1}{x} \cdot \frac{1}{\sqrt{1 + x^2}} \, dx = \int \cot \theta \cos \theta \sec^2 \theta \, d\theta \]
\[ = \int \frac{\cos \theta}{\sin \theta} \cdot \cos \theta \cdot \frac{1}{\cos^2 \theta} \, d\theta = \int \frac{1}{\sin \theta} \, d\theta \]
\[ = \int \csc \theta \, d\theta \]
\[ = -\ln | \csc \theta + \cot \theta | + C \]
\[ = -\ln | \frac{1}{x} \sqrt{1 + x^2} + \frac{1}{x} | + C = -\ln | \frac{1 + \sqrt{1 + x^2}}{x} | + C \]

Ex: \[ \int \frac{x^2}{\sqrt{1 + x^2}} \, dx \]

\[ \tan \theta = x \Rightarrow \sec^2 \theta = \frac{d\theta}{x} \]
\[ \sec^2 \theta = \tan^2 \theta \]
\[ \cos \theta = \frac{1}{\sqrt{1 + x^2}} \]

\[ \int \frac{x^2}{\sqrt{1 + x^2}} \, dx = \int x^2 \left( \frac{1}{\sqrt{1 + x^2}} \right) \, dx = \int \tan^2 \theta \cos \theta \sec \theta \, d\theta = \int \tan^2 \theta \sec \theta \, d\theta \]
\[ \int \sec^2 \theta \, d\theta - \int \sec \theta \, d\theta = \int (\sec^2 \theta - 1) \sec \theta \, d\theta = \int \tan^2 \theta \sec \theta \, d\theta = \int \tan \theta (\sec \theta + \tan \theta) \, d\theta \]
\[ = \sec \theta + \tan \theta - \int \sec^2 \theta \, d\theta \]

* Then solve for \[ \int \sec^3 \theta \, d\theta \], re-substitute into \[ \sec \theta + \tan \theta - \int \sec^3 \theta \, d\theta \] for \( x \).
Partial Fraction Decomposition

Idea: factor the denominator, and split the fraction up into additive parts so that the numerators have degree 1 less than the denominators. For instance:

\[
\begin{align*}
\text{Original Function} & \quad \text{Decomposes like} \\
\frac{1}{(x-1)(x+1)} & \quad \frac{A}{x-1} + \frac{B}{x+1} \\
\frac{1}{(x^2+1)(x-1)} & \quad \frac{Ax+B}{x^2+1} + \frac{C}{x-1} \\
\frac{1}{(x^3+x+1)(x^2+1)(x+1)} & \quad \frac{Ax^2+Bx+C}{x^3+x+1} + \frac{Dx+E}{x^2+1} + \frac{F}{x+1}
\end{align*}
\]

In the event that a root is repeated, make a new summand for each degree leading up to the number of repetitions of the root. For instance:

\[
\begin{align*}
\text{Original function} & \quad \text{Decomposes like} \\
\frac{1}{(x-1)^3(x+1)} & \quad \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\
\frac{1}{(x-1)^3(x+1)} & \quad \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} \\
\frac{1}{(x^3+1)^2(x+1)} & \quad \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+1}
\end{align*}
\]
In the context of integration, this technique can be used to make integration easier or to make it possible for a given function.

Ex: \( \int \frac{1}{x^2 - 1} \, dx \)

Well, \( \frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} \). Set \( \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \).

Then \( A(x+1) + B(x-1) = 1 \). Letting \( x = 1 \) gives \( A = \frac{1}{2} \). Letting \( x = -1 \) gives \( B = -\frac{1}{2} \). So \( \frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \).

Therefore,
\[
\int \frac{1}{x^2 - 1} \, dx = \frac{1}{2} \int \frac{1}{x-1} - \frac{1}{x+1} \, dx = \frac{1}{2} \left( \ln |x-1| - \ln |x+1| \right) + C
= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C.
\]

Ex: \( \int \frac{dx}{(x^2 + 1)(x+1)} \)

After PFD, \( \int \frac{dx}{(x^2 + 1)(x+1)} = \int \frac{-1/2 x + 1/2}{x^2 + 1} + \frac{1/2}{x+1} \, dx = \frac{1}{2} \int \frac{1-x}{x^2 + 1} + \frac{1}{x+1} \, dx \)
\[
= \frac{1}{2} \int \frac{-1}{x^2 + 1} + \frac{1}{x+1} \, dx = \frac{1}{2} \int \frac{-1}{2} \left( \frac{2x}{x^2 + 1} \right) + \frac{1}{x^2 + 1} + \frac{1}{x+1} \, dx \]
\[
= \frac{1}{2} \left( -\frac{1}{2} \ln (x^2 + 1) + \arctan(x) + \ln |x+1| \right) + C.
\]
Improper Integrals

Idea: Integrating over discontinuities or off to infinity. To be technically accurate, we will need to combine integration techniques with limit techniques.

\[
\int_{-\infty}^{\infty} \frac{1}{x^2} \, dx
\]

\[
\int_{-\infty}^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_{-b}^{b} \frac{1}{x^2} \, dx = \lim_{b \to \infty} -\frac{1}{x} \bigg|_{-b}^{b}
\]

\[
= \lim_{b \to \infty} \left( \frac{1}{b} - \frac{1}{b} \right) = \lim_{b \to \infty} (1 - \frac{1}{b}) = 1 - 0 = 1.
\]

Ex: \[ \int_{0}^{\infty} xe^{-x} \, dx \]

\[ u = x \quad du = e^{-x} \, dx \quad \text{gives} \quad \int xe^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C. \]

\[ du = dx \quad v = -e^{-x} \]

Hence,

\[ \int_{0}^{\infty} xe^{-x} \, dx = \lim_{b \to \infty} \int_{0}^{b} xe^{-x} \, dx = \lim_{b \to \infty} \left( -xe^{-x} - e^{-x} \right) \bigg|_{0}^{b} \]

\[ = \lim_{b \to \infty} \left( xe^{-x} + e^{-x} \right) \bigg|_{0}^{b} = 1 - \lim_{b \to \infty} \left( \frac{b}{e^b} + \frac{1}{e^b} \right) \bigg|_{0}^{b} = 1 - \lim_{b \to \infty} \left( \frac{1}{e^b} \right) = 1. \]
Sequences

Idea: You have an infinite set of numbers that may or may not converge to a single number at infinity. To take such a limit, you will use limit notation, restricting the variable over which the limit is being taken to the integers.

Note: If the limit of \( f(x) \) exists and is \( L \), then the limit of the sequence \( \{ f(n) \} \ (n = 1, 2, 3, \ldots) \) exists and is also \( L \). The converse does not hold in general.

Integral Test

Idea: If \( \int_a^\infty f(x) \, dx \) converges, then \( \sum_{n=a}^\infty f(n) \) converges (not necessarily to the same value). Also, if \( \int_a^\infty f(x) \, dx \) diverges, then \( \sum_{n=a}^\infty f(n) \) diverges.

Ex: \( \sum_{n=1}^\infty \frac{1}{n} \)

\[
\int_1^\infty \frac{1}{x} \, dx = \ln|x| \bigg|_1^\infty = \infty - 1 \text{, diverges} \Rightarrow \sum_{n=1}^\infty \frac{1}{n} \text{ diverges.}
\]
Direct Comparison Test

Idea: A series that is less than a convergent series (for sufficiently large $n$) is convergent. A series that is greater than a divergent series (for sufficiently large $n$) is divergent. The preceding statements both hold so long as the terms of the series are positive.

Ex: \[ \sum_{n=1}^{\infty} \frac{n-1}{n^2 \sqrt{n}} \]

Since \[ \frac{n-1}{n^2 \sqrt{n}} \leq \frac{1}{n \sqrt{n}} = \frac{1}{n^{3/2}} \]
and \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges by p-series}, \]
\[ \sum_{n=1}^{\infty} \frac{n-1}{n^2 \sqrt{n}} \text{ converges}. \]

Ex: \[ \sum_{n=1}^{\infty} \left( \frac{\sin n}{n} \right)^2 \]

Since \[ \left( \frac{\sin n}{n} \right)^2 = \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2} \]
and \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges by p-series}, \]
\[ \sum_{n=1}^{\infty} \left( \frac{\sin n}{n} \right)^2 \text{ converges}. \]

Ex: \[ \sum_{n=1}^{\infty} \frac{1}{n!} \]

Since \[ \frac{1}{n!} \leq \frac{1}{2^n} \]
and \[ \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges by geometric series}, \]
\[ \sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges}. \]
**Limit Comparison Test**

Idea: If \( \sum a_n \) and \( \sum b_n \) are series with positive terms and

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0,
\]
then both series converge or both series diverge.

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**Ex:** \( \sum_{n=q}^{\infty} \frac{1}{n^{3/2} - 2n - 6} \)

Consider \( \sum_{n=q}^{\infty} \frac{1}{n^{3/2}} \), which converges by \( p \)-series. Then we have:

\[
\lim_{n \to \infty} \left( \frac{1}{n^{3/2} - 2n - 6} \right) = \lim_{n \to \infty} \frac{n^{3/2}}{n^{3/2} - 2n - 6} = \lim_{n \to \infty} \frac{1}{1 - 2n^{-1/2} - 6n^{-3/2}} = 1 > 0.
\]

Therefore, \( \sum_{n=q}^{\infty} \frac{1}{n^{3/2} - 2n - 6} \) converges.

---

**Ex:** \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \)

Consider \( \sum_{n=1}^{\infty} \frac{1}{n} \), which diverges by \( p \)-series. Then we have

\[
\lim_{n \to \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{k \to 0} \frac{\sin(k)}{k} = \lim_{k \to 0} \cos(k) = 1 > 0.
\]

Therefore, \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) diverges.
Alternating Series Test

Idea: If a sum is alternating and the terms have limit zero (and are decreasing), the series converges.

Ex: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges.

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Ratio Test

Idea: A series \( \sum a_n \) converges if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \) and diverges if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \). The test is indeterminate otherwise. 

Note: use this test if you see factorials!

Ex: \( \sum_{n=1}^{\infty} \frac{n!}{100^n} \)

\[ \lim_{n \to \infty} \left| \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} \right| = \lim_{n \to \infty} \left| \frac{n+1}{100} \right| = \infty > 1. \] So \( \sum_{n=1}^{\infty} \frac{n!}{100^n} \) diverges.

Ex: \( \sum_{n=1}^{\infty} \frac{n^3}{3^n} \)

\[ \lim_{n \to \infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^3}{3n^3} \right| = \frac{1}{3} < 1, \] so \( \sum_{n=1}^{\infty} \frac{n^3}{3^n} \) converges.
Root Test

Idea: A series $\sum a_n$ converges if $\lim_{n \to \infty} \sqrt[n]{|a_n|} \leq 1$ and diverges if $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1$. The test is inconclusive otherwise.

Ex: $\sum_{n=1}^{\infty} \left( \frac{n^3+1}{2n^2+1} \right)^n$

$$\lim_{n \to \infty} \sqrt[n]{\left( \frac{n^3+1}{2n^2+1} \right)^n} = \lim_{n \to \infty} \frac{n^3+1}{2n^2+1} = \frac{1}{e} < 1,$$ so $\sum_{n=1}^{\infty} \left( \frac{n^3+1}{2n^2+1} \right)^n$ converges.

Power Series

Idea: Turn a regular function into an infinite sum.

Ex: $\frac{1}{1-x}$.

Doing long division gives $1 - x + x^2 - \frac{x^3}{1-x} + \frac{2x^4}{(1-x)^2} - \frac{6x^5}{(1-x)^3} + \ldots$

Hence, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$.
Taylor Series

Idea: By picking a center $c$, we may write any function (in some radius of convergence about $c$) as an infinite sum using various ordered derivatives of the function evaluated at $c$.

The Taylor Series of $f(x)$ at $C$ is \[ \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \, . \]

Note: Power Series are a special case of Taylor Series in which finding the derivatives isn't necessary.

Double note: Much of the time, you can make your life easier by finding the Taylor series of a simpler function and then substituting for $x$.

\[ \text{Ex}: \text{ Find the Taylor Series and radius of convergence for } \arctan x, \text{ at } c = 0. \]

Note that $\arctan x = \int_{0}^{x} \frac{1}{1+t^2} \, dt$. Therefore, finding the Taylor Series for $\frac{1}{1+t^2}$ will put us in good shape. However, we already knew the power series for $\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$. Since this power series is already at center $c = 0$, we need only integrate to find the power series (and hence the Taylor Series) of $\arctan x$.

$\arctan x = \int_{0}^{x} \frac{1}{1+t^2} \, dt = \int_{0}^{x} \sum_{n=0}^{\infty} (-1)^n t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

\[ \downarrow \text{cont'd.} \]
The radius of convergence can be obtained (in part) by the Ratio Test.

\[
\lim_{n \to \infty} \frac{(-1)^n x^{n+1}}{(n+1)!} \cdot \frac{\partial^{n+1}}{(-1)^n x^{n+1}} = \lim_{n \to \infty} \frac{(\partial^{n+1}) x^{n+3}}{2n+3}
\]

\[
= x^2.
\]

\[x^2 < 1 \implies -1 < x < 1\]

Now we need to check the endpoints.

\[x = 1: \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)!}, \text{ converges by AST.}\]

\[x = -1: \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{-1}{(n+1)!}, \text{ converges by AST.}\]

Hence, the interval of convergence of the Taylor series is \([-1, 1]\), with radius 1.
Ex: \( f(x) = e^{-x^3} + \cos x \)

Note: \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \). These can be verified by applying the definition for the Taylor series to each.

Since \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), \( e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \). Therefore,

\[
e^{-x^3} + \cos x = \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) + \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{n!} x^{2n} + \frac{(-1)^n}{(2n)!} x^{2n} \right)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{n!} + \frac{1}{(2n)!} \right) x^{2n}
\]

We apply the ratio test for the interval of convergence:

\[
\lim_{n \to \infty} \left| \frac{(\frac{1}{n!} + \frac{1}{(2n)!}) x^{2(n+1)}}{\frac{1}{n!} + \frac{1}{(2n)!}} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{(n+1)!} + \frac{1}{(2(n+1))!}}{\frac{1}{n!} + \frac{1}{(2n)!}} \right| x^2
\]

\[
= \lim_{n \to \infty} \left| \frac{(2n+2)! + (n+1)!}{(2n)! + n!} \cdot \frac{n! (2n)!}{(n+1)! (2n+2)!} \right| x^2
\]

\[
= \lim_{n \to \infty} \left| \frac{(2n+2)! + (n+1)!}{(n+1)(2n+1)(2n)!} \cdot \frac{n! (2n)!}{(n+1)! (2n+2)!} \right| x^2
\]

\[
= \lim_{n \to \infty} \left| \frac{\deg 2n+3 \text{ poly}}{\deg 2n+3 \text{ poly}} \right| x^2
\]

\[
= 0 < 1 \quad \forall x. \quad \text{Therefore, the IOC is } (-\infty, \infty).
\]
Ex: Calculate $e^{-0.2}$ correct to 5 decimal places.

We apply the Taylor Remainder Theorem. The error in the $n^{th}$ Taylor polynomial is less than or equal to

$$\max_{x_i, x_3 \in I} \left| \frac{f^{(n+1)}(x_i)}{(n+1)!} (x_3 - c)^{n+1} \right|$$

where $I$ is the interval under consideration and $c$ is the center.

In this case, the interval $[-0.3, 0]$ will suffice. Since $\frac{d^n}{dx^n} e^x = e^x$ for any $n$, we have

$$\text{Error} \leq \max_{x_i, x_3 \in [-0.3, 0]} \left| \frac{e^{x_i}}{(n+1)!} (x_3 - 0)^{n+1} \right|$$

Further, since $e^x$ increases on its whole domain, the error may be maximized as follows:

$$\text{Error} \leq \max_{x_i, x_3 \in [-0.3, 0]} \left| \frac{e^{x_i}}{(n+1)!} (x_3)^{n+1} \right| = \left| \frac{e^0}{(n+1)!} (0.2)^{n+1} \right| = \frac{0.2^{n+1}}{(n+1)!}$$

Plugging in $n$, until $\frac{0.2^{n+1}}{(n+1)!} \leq 0.00001$ will finish it off.
Area between curves

\[ \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \]

or written more simply, \[ \int_a^b f(x) - g(x) \, dx \].

Volumes of Revolution: Disk method

Idea: rotating \( f(x) \) about an axis, we seek the volume of the resulting solid.

\[ \text{Volume} = \pi \left( f(x) \right)^2 \, dx \]

By summing over an infinite number of such \( dx \)-disks, we obtain our formula: \[ \pi \int_a^b (f(x))^2 \, dx \] (assuming the axis of rotation is the \( x \)-axis).
Ex: (Gabriel’s Horn) Rotate $\frac{1}{x}$ about the $x$-axis from $x=1$ to $\infty$.

\[ \pi \int_{1}^{\infty} \left(\frac{1}{x}\right)^2 \, dx = \pi \int_{1}^{\infty} \frac{1}{x^2} \, dx \]

\[ = \pi \int_{1}^{\infty} x^{-2} \, dx = \pi \left( -x^{-1} \bigg|_{1}^{\infty} \right) \]

\[ = \pi \left( x^{-1} \bigg|_{1}^{\infty} \right) = \pi (1 - 0) = \pi \]

Volumes of Revolution: Shell method

Idea: Same as first section, but now we will piece the volume together in a different way.

Summary over an infinite number of such $dx$-shells, we obtain the formula:

\[ \pi \int_{a}^{b} x f(x) \, dx \] (assuming the axis of rotation is the $y$-axis)
Ex: Rotate $\sqrt{x}$ about the $y$-axis from $x = 0$ to $x = 1$.

\[
2\pi \int_0^1 x \sqrt{x} \, dx = 2\pi \int_0^1 x^{3/2} \, dx
\]

\[
= 2\pi \left[ \frac{x^{5/2}}{5/2} \right]_0^1
\]

\[
= 2\pi \left( \frac{2}{5} \cdot 32 \right) = \frac{128}{5} \pi
\]

**Polar Coordinates**

Idea: coordinateize the plane by using radius and angle.

Ex:

\[
(2, 2\sqrt{3}) \quad \text{put } (2, 2\sqrt{3}) \text{ into polar form.}
\]

We obtain a triangle

It happens to be a 30-60-90 triangle, so $\Theta = 60^\circ = \pi/3$ rad,

and $r = 4$.

Hence, $(2, 2\sqrt{3}) \rightarrow (4, \pi/3)$ in polar.
Polar area

Integration in polar will yield the area of the space between a polar function's edge and the origin over a theta-range, as below. However, correction factors need to be made to account for the circular method in which we are integrating.

\[
\text{Area between } \theta_1 \text{ and } \theta_2 = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.
\]

Note of warning: polar functions really, REALLY like to cancel out their own areas. You will likely need to break up your integral to obtain the actual area of a region, especially if the function in question is rose-type. Use the symmetries of the function to your advantage to make sure your solution is correct.

In the case of a rose: find the area of one petal and then multiply by the number of petals.