INSTRUCTIONS:

(1) To receive credit you must:
   (a) work in a logical fashion, show all your work, indicate your reasoning;
       no credit will be given for an answer that just appears;
       such explanations help with partial credit
   (b) if a line/box is provided, then:
       — show you work BELOW the line/box
       — put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points.
    Check that your copy of the exam has all of the problems.

(3) You may not use a calculator, books, personal notes.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When
    you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from Calculus by Stewart, 6th ed., ET):
    11.2–11.8.

**Problem Inspiration:** Mostly homework and old exam problems. See the solution key for details.
1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. $n^{th}$-term test for an arbitrary series $\sum a_n$.
If $\lim_{n \to \infty} a_n \neq 0$ or $\lim_{n \to \infty} a_n$ does not exist, then $\sum a_n$ diverges.

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$
- converges if and only if $|r| < 1$
- diverges if and only if $|r| \geq 1$

1c. $p$-series where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$
- converges if and only if $p > 1$
- diverges if and only if $p \leq 1$

1d. Integral Test for a positive-terminated series $\sum a_n$ where $a_n \geq 0$.
Let $f : [1, \infty) \to \mathbb{R}$ be so that
- $a_n = f\left(\frac{1}{n}\right)$ for each $n \in \mathbb{N}$
- $f$ is a positive, increasing function
- $f$ is a decreasing function
Then $\sum a_n$ converges if and only if $\int_{1}^{\infty} f(x) \, dx$ converges.

1e. Comparison Test for a positive-terminated series $\sum a_n$ where $a_n \geq 0$.
- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-terminated series $\sum a_n$ where $a_n \geq 0$.
Let $b_n > 0$ and $\lim_{n \to \infty} \frac{b_n}{a_n} = L$.
If $0 < L < \infty$, then $\sum a_n$ converges.

1g. Ratio and Root Tests for a positive-terminated series $\sum a_n$ where $a_n \geq 0$.
Let $\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n \to \infty} (a_n)^{\frac{1}{n}}$.
- If $\rho < 1$, then $\sum a_n$ converges.
- If $\rho > 1$, then $\sum a_n$ diverges.
- If $\rho = 1$, then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum (-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.
If
- $a_n > a_{n+1}$ for each $n \in \mathbb{N}$
- $\lim_{n \to \infty} a_n = 0$
then $\sum(-1)^n a_n$ converges.
1. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ __converges__
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ __converges__ and $\sum |a_n|$ __diverges__
- $\sum a_n$ is divergent if and only if $\sum a_n$ __diverges__

1j. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=1}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

| Does $\sum |a_n|$ converge? |
|---------------------------|
| Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test. |
| **if NO** |
| **if YES** |
| **if NO** |
| **if YES** |

| Does $\lim_{n \to \infty} |a_n| = 0$? |
|-------------------------------------|
| **if NO** |
| **if YES** |

<table>
<thead>
<tr>
<th>$\sum a_n$ is absolutely convergent</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Is $\sum a_n$ an alternating series?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>if YES</strong></td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>if YES</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sum a_n$ is conditionally convergent</th>
</tr>
</thead>
</table>

2. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

| T | F | If $\lim_{n \to \infty} a_n = 0$, then $\sum a_n$ converges |
| T | F | If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. |
| T | F | If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges. |
| T | F | If $(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge. |
| T | F | If $S_N = \sum_{n=1}^{N} r^n$, then $S_N = \frac{\frac{r - r^{N+1}}{1-r}, r \neq 1}{1 - r}$. |
3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

\[ \sum_{n=17}^{\infty} \frac{(-1)^n}{n} \]

- Absolutely convergent
- Conditionally convergent
- Divergent

Check for A.C.

\[ \sum_{r=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{r=1}^{\infty} \frac{1}{n} \]

\[ A_{n+1} = \frac{1}{n} > \frac{1}{n+1} = a_{n+1} \]

\[ \therefore \lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{so by A.S.T} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges} \]
4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}} \]

\[ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)(k+2)}} = \sum \frac{1}{(k+1)^{3/2}} = a_n \]

\[ \text{LCT} \quad b_n = \left( \frac{n^3}{2} \right)^{1/2} \]

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n^3}{2} \right)^{1/2} = 1 \text{ so } 0 < L < 1 \text{ so } \sum a_n \text{ diverges.} \]

In the same way, \( \sum b_n = \sum \frac{1}{(n^3)^{1/2}} = \frac{1}{n^{3/2}} \text{ p-series, with } p = \frac{3}{2} > 1 \text{ so \( \sum b_n \text{ converges.} \) Since \( \sum b_n \text{ converges, } \sum \frac{1}{\sqrt{k(k+1)(k+2)}} \text{ converges by the LCT.} \]
5. Let
\[ a_n = \frac{n!}{(2n - 1)!} \]

5a. Find an expression for \( \frac{a_{n+1}}{a_n} \) that does NOT have a factorial sign (that is a ! sign) in it.

\[ \frac{a_{n+1}}{a_n} = \frac{n+1}{2n+1} \]

5b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n - 1)!} \]

- Absolutely convergent
- Conditionally convergent
- Divergent

**Ratio Test:**

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2(n+1)!} \cdot \frac{2n+1}{n!} = \frac{2n+1}{2n+1} \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+1}{2n+1} = 0 < 1 \]

So, by the **Ratio Test**, the series converges.
6. Consider the formal power series
\[ \sum_{n=1}^{\infty} \frac{(5x + 10)^n}{n} \]

Hint: \((5x + 10)^n = [5(x + 2)]^n = 5^n (x + 2)^n = 5^n (x - (-2))^n\)

The center is \(x_0 = \frac{-2}{5}\) and the radius of convergence is \(R = \frac{1}{5}\).

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

\[\begin{array}{c}
\text{diverge} \downarrow \\
-\frac{11}{5} \quad -2 \quad -\frac{9}{5} \\
\text{convergent} \downarrow \\
-\frac{9}{5} \quad \frac{9}{5} \\
\text{diverge} \downarrow \\
\end{array}\]

\[\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{5(x+2)^{n+1}}{5(x+2)^n + 1} \right| = \left| 5(x+2) \right| \lim_{n \to \infty} \frac{n}{n+1} = \left| 5(x+2) \right| \frac{1}{\frac{n+1}{n}} = \left| 5(x+2) \right| \frac{n}{n+1} \]

For \(5 \left| x + \frac{2}{5} \right| < 1 \Rightarrow \left| x + \frac{2}{5} \right| < \frac{1}{5}\)

Test endpoints:
\[x = -\frac{9}{5}, \quad x = \frac{9}{5}\]

\[\sum \frac{(1)^n}{n} = \sum \frac{1}{n} \quad \text{p-series diverges with } p = 1\]

\[\sum \frac{(-1)^n}{n} \quad \text{nice} \]

Check for A.C.:
\[\sum \frac{1}{n} \quad \text{diverges} \]
\[\lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{so by A.C.} \sum \frac{(-1)^n}{n} \quad \text{converges} \]
7. Consider the formal power series

$$\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}.$$ 

Hint 1: \( \frac{x^n}{(\ln n)^n} = \left[ \frac{x}{\ln n} \right]^n \) so would you rather use the root test or the ratio test?

Hint 2: \( \ln(a^r) = r \ln(a) \) but \( (\ln(a))^r \neq r \ln(a) + \)

The center is \( x_0 = \) and the radius of convergence is \( R = \).

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

Absolutely convergent for all \( x \)

Root test:

$$\lim_{n \to \infty} \left[ \frac{x}{(\ln n)^n} \right] = \lim_{n \to \infty} \frac{x}{\ln n} = x \lim_{n \to \infty} \frac{1}{\ln n} = x \cdot 0 = 0$$

so convergent for all \( x \).
8. Fill-in the 6 blanks.
Consider the power series
\[ \sum_{n=1}^{\infty} (-1)^n a_n x^n \]
where all of the \( a_n \)'s are positive. Let's say that you know that

\[
\begin{align*}
&\text{if } 0 < x < 17 \quad \text{then } \sum (-1)^n a_n x^n \text{ converges} \\
&\text{if } x = 17 \quad \text{then } \sum (-1)^n a_n x^n \text{ conditionally converges} \\
&\text{if } 17 < x \quad \text{then } \sum (-1)^n a_n x^n \text{ diverges}.
\end{align*}
\]

Then this power series has:

- Center at \( x_0 = 0 \) and radius of convergence \( R = 17 \).

Also, what can you say about the following interval? Fill in the blanks below with:

- is absolutely convergent
- is conditionally convergent
- is divergent
- inconclusive (not enough information given to decide in general).

\[
\begin{align*}
&\text{if } -17 < x < 0 \quad \text{then } \sum (-1)^n a_n x^n \quad \text{absolutely convergent} \\
&\text{if } x < -17 \quad \text{then } \sum (-1)^n a_n x^n \quad \text{divergent} \\
&\text{if } x = 0 \quad \text{then } \sum (-1)^n a_n x^n \quad \text{absolutely convergent} \\
&\text{if } x = -17 \quad \text{then } \sum (-1)^n a_n x^n \quad \text{divergent} \\
\sum (-1)^n a_r \text{ for } r \text{ odd} \quad \text{conditionally convergent} \\
\sum (-1)^n a_r \text{ for } r \text{ even} \quad \text{divergent}
\end{align*}
\]
9. Geometric Series. Let, for $N \geq 102$,

$$s_N = \sum_{n=102}^{N} 3 \cdot \frac{2^n}{7^{n+1}}.$$

9a. Do some algebra to write $s_N$ as $\sum_{n=102}^{N} c \cdot r^n$ for an appropriate constant $c$ and ratio $r$.

$$s_N = \sum_{n=102}^{N} \frac{3}{7} \left( \frac{2}{7} \right)^n$$

$$3 \sum \frac{2^n}{7^n} = \frac{3}{7} \sum \frac{2^n}{7^n} = \sum \frac{3}{7} \left( \frac{2}{7} \right)^n$$

9b. Using the method from class (rather than some formula), find an expression for $s_N$ in closed form (i.e. without a summation $\sum$ sign nor some dots $\ldots$).

$$s_N = \frac{3}{5} \left[ \left( \frac{2}{7} \right)^{102} - \left( \frac{2}{7} \right)^{N+1} \right]$$

$$s_N = \frac{3}{7} \left( \frac{2}{7} \right)^{102} + \frac{3}{7} \left( \frac{2}{7} \right)^{103} + \ldots + \frac{3}{7} \left( \frac{2}{7} \right)^{N}$$

$$- \left( \frac{2}{7} \right)^{102}$$

$$s_N = \frac{3}{7} \left[ \left( \frac{2}{7} \right)^{102} - \left( \frac{2}{7} \right)^{N+1} \right]$$

$$s_N = \frac{3}{7} \left[ \left( \frac{2}{7} \right)^{102} - \left( \frac{2}{7} \right)^{N+1} \right]$$

9c. Does $\sum_{n=102}^{\infty} 3 \cdot \frac{2^n}{7^{n+1}}$ converge or diverge? If it converges, find its sum. Justify your answer.

$$\sum_{n=102}^{\infty} 3 \cdot \frac{2^n}{7^{n+1}} \text{ converges to } \frac{3}{5} \left( \frac{2}{7} \right)^{102}$$

$$\sum_{n=102}^{\infty} 3 \cdot \frac{2^n}{7^{n+1}} \text{ is a geometric series with } |r| = \frac{2}{7} < 1 \text{ so } \sum_{n=102}^{\infty} 3 \cdot \frac{2^n}{7^{n+1}} \text{ converges}$$

$$s_N = \frac{3}{5} \left[ \left( \frac{2}{7} \right)^{102} - \left( \frac{2}{7} \right)^{N+1} \right]$$

$$\lim_{r \to -\infty} s_N = \frac{3}{5} \left( \frac{2}{7} \right)^{102}$$