MARK BOX

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 a-y</td>
<td>25</td>
</tr>
<tr>
<td>2 a-o</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4 a-c</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>POSSIBLE</td>
<td>108</td>
</tr>
</tbody>
</table>

INSTRUCTIONS:

1. To receive credit you must:
   a. work in a logical fashion, show all your work, indicate your reasoning;
      no credit will be given for an answer that just appears;
      such explanations help with partial credit
   b. if a line/box is provided, then:
      — show you work BELOW the line/box
      — put your answer on/in the line/box
   c. if no such line/box is provided, then box your answer
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. You may not use a calculator, books, personal notes.
4. During this exam, do not leave your seat. If you have a question, raise your hand. When
   you finish: turn your exam over, put your pencil down, and raise your hand.
5. This exam covers (from Calculus by Anton, Bivens, Davis 8th ed.):
   S 7.1 - 7.4, 7.6, 7.7, 8.1 - 8.5, 8.8, 10.1-10.10, 11.1 - 11.3...
1. Fill in the blanks (each worth 1 point).

1a. \( \int \frac{du}{u} = \ln |u| + C \)

1b. If \( a \) is a constant and \( a > 0 \) but \( a \neq 1 \), then \( \int a^u \, du = \frac{a^u}{\ln a} + C \)

1c. \( \int \cos u \, du = \sin u + C \)

1d. \( \int \sec^2 u \, du = \tan u + C \)

1e. \( \int \sec u \tan u \, du = \sec u + C \)

1f. \( \int \sin u \, du = -\cos u + C \)

1g. \( \int \csc^2 u \, du = -\cot u + C \)

1h. \( \int \csc u \cot u \, du = -\csc u + C \)

1i. \( \int \tan u \, du = -\ln |\cos u| + C \)

1j. \( \int \cot u \, du = \ln |\sin u| + C \)

1k. \( \int \sec u \, du = \ln |\sec u + \tan u| + C \)

1l. \( \int \csc u \, du = -\ln |\csc u + \cot u| + C \)

1m. If \( a \) is a constant and \( a > 0 \) then \( \int \frac{1}{\sqrt{a^2-u^2}} \, du = \frac{1}{a} \sin^{-1} \frac{u}{a} + C \)

1n. If \( a \) is a constant and \( a > 0 \) then \( \int \frac{1}{a^2+u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C \)

1o. If \( a \) is a constant and \( a > 0 \) then \( \int \frac{1}{u\sqrt{a^2-u^2}} \, du = \frac{1}{a} \sec^{-1} \frac{u}{a} + C \)

1p. Partial Fraction Decomposition. If one wants to integrate \( \frac{f(x)}{g(x)} \) where \( f \) and \( g \) are polynomials and \([\text{degree of } f] \geq [\text{degree of } g]\), then one must first do **long division**.

1q. Integration by parts formula: \( \int u \, dv = uv - \int v \, du \)

1r. Trig substitution: (recall that the integrand is the function you are integrating)
   - if the integrand involves \( a^2-u^2 \), then one makes the substitution \( u = a \sin \theta \)
   - if the integrand involves \( a^2+u^2 \), then one makes the substitution \( u = a \tan \theta \)
   - if the integrand involves \( u^2-a^2 \), then one makes the substitution \( u = a \sec \theta \)

1u. trig formula ... your answer should involve trig functions of \( \theta \), and not of \( 2\theta \): \( \sin(2\theta) = 2\sin \theta \cos \theta \)

1v. trig formula ... \( \cos(2\theta) \) should appear in the numerator: \( \cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \)

1w. trig formula ... \( \cos(2\theta) \) should appear in the numerator: \( \sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \)

1x. trig formula ... since \( \cos^2 \theta + \sin^2 \theta = 1 \), we know that the corresponding relationship between tangent (i.e., \( \tan \)) and secant (i.e., \( \sec \)) is \( \sec^2 \theta - \tan^2 \theta = 1 \)

1y. \( \arctan \left( \frac{1}{\sqrt{3}} \right) = \frac{-\pi}{6} \) **RADIANS**. (your answer should be an angle)

\[
\tan = \frac{\sin}{\cos} \quad \tan \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{3} \quad \tan \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{3}
\]
2. Fill-in-the-blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

2a. Sequences Let $-\infty < r < \infty$. (Fill-in-the blanks with exists or does not exist, i.e. DNE)

- $0$ • If $|r| < 1$, then $\lim_{n \to \infty} r^n$ exists __________
- $\infty$ • If $|r| > 1$, then $\lim_{n \to \infty} r^n$ does not exist __________
- $1$ • If $r = 1$, then $\lim_{n \to \infty} r^n$ exists __________
- $\infty$ • If $r = -1$, then $\lim_{n \to \infty} r^n$ does not exist __________

2b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r| < 1$ __________
- diverges if and only if $|r| \geq 1$ __________

2c. $p$-series where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if $p > 1$ __________
- diverges if and only if $p \leq 1$ __________

2d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f : [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f(\frac{1}{n})$ for each $n \in \mathbb{N}$
- $f$ is a positive function
- $f$ is a decreasing function
- $f$ is a continuous function

Then $\sum a_n$ converges if and only if $\int f(x) \, dx$ converges.

2e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges __________
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges __________

2f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n \to \infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges __________

2g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n \to \infty} (a_n)^{\frac{1}{n}}$.

- If $\rho < 1$ then $\sum a_n$ converges.
- If $\rho > 1$ then $\sum a_n$ diverges.
- If $\rho = 1$ then the test is inconclusive.

2h. Alternating Series Test for an alternating series $\sum (-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If
- $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$ decreasing
- $\lim_{n \to \infty} a_n = 0$

then $\sum (-1)^n a_n$ converges __________

2i. $n^{th}$-term test for an arbitrary series $\sum a_n$.

If $\lim_{n \to \infty} a_n \neq 0$ or $\lim_{n \to \infty} a_n$ does not exist, then $\sum a_n$ diverges __________
2j. By definition, for an arbitrary series \( \sum a_n \), (fill in the blanks with converges or diverges).
   - \( \sum a_n \) is absolutely convergent if and only if \( \sum |a_n| \) converges.
   - \( \sum a_n \) is conditionally convergent if and only if \( \sum a_n \) converges and \( \sum |a_n| \) diverges.
   - \( \sum a_n \) is divergent if and only if \( \sum a_n \) diverges.

2k. Consider a function \( y = f(x) \) where \( f: [1, \infty) \to \mathbb{R} \).

Next consider the corresponding sequence \( \{a_n\}_{n=1}^{\infty} \) where \( a_n \) is defined as \( f(n) \).

   - If the limit of the function \( y = f(x) \) as \( x \to \infty \) is \( L \),
     
     then the limit of the corresponding sequence \( \{a_n\}_{n=1}^{\infty} \) as \( n \to \infty \) is \( L \).
     
   - If \( \lim_{n \to \infty} a_n = L \), is it necessarily true that \( \lim_{x \to \infty} f(x) = L \)? Circle: Yes or No.

For 2l - 2o:

Let \( y = f(x) \) be a function with derivatives of all orders in an interval \( I \) containing \( x_0 \).

Let \( y = p_N(x) \) be the \( N^{th} \)-order Taylor polynomial of \( y = f(x) \) about \( x_0 \).

Let \( y = R_N(x) \) be the \( N^{th} \)-order Taylor remainder of \( y = f(x) \) about \( x_0 \).

Let \( y = p_\infty(x) \) be the Taylor series of \( y = f(x) \) about \( x_0 \).

2l. In open form (i.e., with and without a \( \sum \)-sign)

\[
p_N(x) = f^0(x_0) + f^1(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)(x-x_0)^2}{2!} + \cdots + \frac{f^{(N)}(x_0)(x-x_0)^N}{N!}
\]

2m. In closed form (i.e., with a \( \sum \)-sign and without ...)

\[
p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0) (x-x_0)^n}{n!}
\]

2n. In closed form (i.e., with a \( \sum \)-sign and without ...)

\[
p_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0) (x-x_0)^n}{n!}
\]

2o. We know that \( f(x) = p_N(x) + R_N(x) \). Taylor's BIG Theorem tells us that, for each \( x \in I \),

\[
R_N(x) = \frac{f^{(N+1)}(c)(x-x_0)^{N+1}}{(N+1)!}
\]

for some \( c \) between \( x \) and \( x_0 \).
3. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=1}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge?  
Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

If YES $\Downarrow$

$\sum a_n$ is absolutely convergent

if NO $\Rightarrow$

Does $\lim_{n \to \infty} |a_n| = 0$?  
If NO $\Rightarrow$ $\sum a_n$ is divergent

If YES $\Downarrow$

Is $\sum a_n$ an alternating series?

If YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

If YES $\Downarrow$

$\sum a_n$ is conditionally convergent
4. Let $R$ be the region in the $x-y$-plane that is enclosed by $y = x$ and $y = 4x + 1$ and $x = 0$ and $x = 1$.

In each of problems 4b and 4c:

- $R$ will be revolved around some line to create a solid of revolution
- using either the disk, washer, or shell method, express the volume $V$ of the resulting solid of revolution as one integral (and NOT as 2 or more integrals).
- In the space provided below each problem, make some good enough sketch (does not have to be too fancy) to indicate (i.e., help justify) your thinking/reasoning behind your solution
- you do not have to do lots of algebra to your integrand
- you do not have to integrate your integral.

4a. Make a sketch of $R$ below and label important points. Also, in your sketch of $R$ below, draw in a typical rectangle (should it be horizontal or vertical?) that would be used to express the area of $R$ as precisely 1 integral (and not 2 integrals).

4b. The volume $V$ of the solid obtained by revolving the region $R$ about the $x$-axis is

$$V = \pi \int_{0}^{1} \left( (4x+1)^2 - (x)^2 \right) dx$$

Hint: $(r_{big}^2 - r_{little}^2) \neq (r_{big} - r_{little})^2$
4c. The volume \( V \) of the solid obtained by revolving the region \( R \) about the line \( x = 2 \) is

\[
V = \pi \int_{x=0}^{x=1} ((4x+1-x)(2-x)) \, dx
\]

Hint: the line \( x = 2 \) is a vertical line (it is NOT a horizontal line).

\[2\pi \rightarrow \text{(average height)} \rightarrow \text{(radius)} \rightarrow \text{(thickness)} \]

\[
(4x+1-x) \rightarrow (2-x) \rightarrow \Delta x
\]
\[ V = \pi \int_{0}^{1} \left( (4x+1)^2 - (x)^2 \right) dx \]

\[ V = 2\pi \int_{0}^{1} \text{(ave radius)(height)(thickness)} \]

\[ 2\pi \int_{0}^{1} (4x-1-x)(2-x) \, dx \]
5. \[ \int \sec^3 x \tan^3 x \, dx = \frac{1}{5} \sec^5 x + \frac{-1}{3} \sec^3 x + C \]

Remark: box your substitution box for more partial credit.
Recall: you can check your answer via differentiation (if you have time).

\[ \int \sec^3 x \tan^3 x \, dx = \]
\[ \int (\sec^2 x \tan^2 x \cdot (\sec x \tan x)) \, dx = \]
\[ \int (\sec^2 x \cdot (\sec^2 x - 1) \cdot (\sec x \tan x)) \, dx = \]
\[ \int (\sec^4 x - \sec^2 x) \cdot (\sec x \tan x) \, dx = \]

\[ u = \sec x \]
\[ du = \sec x \tan x \, dx \]

\[ \int (u^4 - u^2) \, du = \]
\[ \frac{1}{5} u^5 - \frac{1}{3} u^3 + \phi \]
\[ \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + \phi \]
\[ \int \frac{dx}{(4 + x^2)^2} = \frac{1}{16} \tan^{-1} \left( \frac{x}{2} \right) + \frac{1}{32} \left( \frac{1}{2} \left( \frac{x}{\sqrt{x^2 + 4}} \right) \left( \frac{2}{\sqrt{x^2 + 1}} \right) \right) + C \]

Remark: box your substitution box for more partial credit.
Recall: you can check your answer via differentiation (if you have time).
Hint: useful might be problems: 1r - 1w.

\[
\int \frac{dx}{(2^2 + x^2)^2} = \frac{u^2 + a^2}{x}
\]

\[\theta = \tan^{-1} \left( \frac{x}{2} \right)\]

\[x = 2 \tan \theta\]

\[\theta = \frac{x}{2}\]

\[\tan \theta = \frac{x}{2}\]

\[
\left( 4 + x^2 \right) = 4 + 4 \tan^2 \theta = 4 (1 + \tan^2 \theta) = 4 \sec^2 \theta
\]

\[
\left( 4 + x^2 \right)^2 = (4 \sec^2 \theta)^2 = 16 \sec^4 \theta
\]

\[
\int \frac{dx}{(4 + x^2)^2} = \int \frac{2 \sec^2 \theta d\theta}{16 \sec^4 \theta}
\]

\[
= \frac{1}{8} \int \frac{d\theta}{\sec^2 \theta}
\]

\[
= \frac{1}{8} \int \cos^2 \theta d\theta
\]

\[
= \frac{1}{8} \int \left( 1 + \cos (2\theta) \right) d\theta
\]

\[
= \frac{1}{8} \left( \frac{1}{2} \int (1 + \cos (2\theta)) \right) d\theta
\]

\[
= \frac{1}{16} \theta + \frac{1}{16} \frac{1}{2} \int \cos u \, du
\]

\[
= \frac{1}{16} \theta + \frac{1}{32} \sin (2\theta) + C
\]

\[
= \frac{1}{16} \tan^{-1} \left( \frac{x}{2} \right) + \frac{1}{32} \left( \frac{2}{\sqrt{x^2 + 1}} \right) \left( \frac{2}{\sqrt{x^2 + 1}} \right) + C
\]
\[
\int \frac{x^3 + x^2 + 2x + 1}{x^4 + 2x^2 + 1} \, dx = \frac{1}{2} \ln |x^2 + 1| + \tan^{-1} x - \frac{1}{2} (x^2 + 1)^{-1} + C
\]

Remark: box your substitution box for more partial credit.
Recall: you can check your answer via differentiation (if you have time).
Hint: \(x^4 + 2x^2 + 1 = (x^2 + 1)^2\), which is an irreducible quadratic squared.

\[
\int \frac{\chi^3 + \chi^2 + 2\chi + 1}{(\chi^2 + 1)^2} \, d\chi = \frac{Ax + B}{(\chi^2 + 1)} + \frac{Cx + D}{(\chi^2 + 1)^2} = \frac{Ax^3 + Ax + Bx^2 + B + Cx + C}{(\chi^2 + 1)^2}
\]

\[
\chi^3: \begin{array}{c}
A = 1 \\
B = 1 \\
A + C = 2 \\
B + D = 1
\end{array}
\]

\[
1 + C = 2 \Rightarrow C = 1
\]

\[
1 + D = 1 \Rightarrow D = 0
\]

\[
\int \frac{Ax + B}{(\chi^2 + 1)} \, d\chi + \int \frac{Cx + D}{(\chi^2 + 1)^2} \, d\chi = \int \frac{\chi + 1}{\chi^2 + 1} \, d\chi + \int \frac{\chi}{(\chi^2 + 1)^2} \, d\chi
\]

\[
= \int \frac{\chi}{\chi^2 + 1} \, d\chi + \int \frac{1}{\chi^2 + 1} \, d\chi + \int \frac{\chi}{(\chi^2 + 1)^2} \, d\chi = \int \frac{1}{u^2} \, du + \int \frac{1}{\chi^2 + 1} \, d\chi + \frac{1}{2} \int \frac{1}{u^2} \, du
\]

\[
= \frac{1}{2} \ln |\chi^2 + 1| + \int \frac{1}{\chi^2 + 1} \, d\chi + \frac{1}{2} \cdot -1 u^{-1} + C
\]

\[
= \frac{1}{2} \ln |\chi^2 + 1| + \tan^{-1} \chi + \frac{1}{2} (\chi^2 + 1)^{-1} + C
\]
\[ \int x^2 e^{x^2} \, dx = \frac{1}{2} e^{x^2} x^2 - \frac{1}{2} e^{x^2} + C \]

Remark: box your substitution box for more partial credit.
Recall: you can check your answer via differentiation (if you have time).
Hint: one lesson from parts was to look for a \( dv \) that is easy to integrate and thus get the \( u \).
\[ \int \sin(\ln x) \, dx = \frac{x \sin(\ln x) - x \cos(\ln x)}{2} + C \]

Remark: box your substitution box for more partial credit.
Recall: you can check your answer via differentiation (if you have time).
Hint: one lesson from parts was that if the integrand is easy to differentiate but hard to integrate, then let \( u \) be in integrand. Another lesson from parts was the bring to the other side idea.

\[ \int \sin(\ln x) \, dx = x \sin(\ln x) - \int \cos(\ln x) \, \frac{1}{x} \, dx \]

\[ u = \sin(\ln x), \quad dv = dx \]
\[ du = \cos(\ln x) \frac{1}{x} \, dx, \quad v = x \]

\[ \int \cos(\ln x) \, \frac{1}{x} \, dx \]
\[ u = \cos(\ln x), \quad dv = dx \]
\[ du = -\sin(\ln x) \frac{1}{x} \, dx, \quad v = x \]

\[ \int \cos(\ln x) \, \frac{1}{x} \, dx = x \cos(\ln x) - \int \frac{x}{x} \sin(\ln x) \, dx \]

\[ \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) - \int \frac{x}{x} \sin(\ln x) \, dx \]

\[ 2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) \]

\[ \int \sin(\ln x) \, dx = \frac{x \sin(\ln x) - x \cos(\ln x)}{2} \]
10. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

\[
\sum_{n=17}^{\infty} (-1)^n \frac{1}{\sqrt{n^2 - 1}}
\]

- absolutely convergent
- conditionally convergent
- divergent

**abs conv?**

\[
\sum |a_n| = \sum \frac{1}{\sqrt{n^2 - 1}} = \sum \frac{1}{(n^2 - 1)^{1/2}} = \sum b_n = \frac{1}{n}
\]

\[
\lim_{n \to \infty} \frac{|a_n|}{b_n} = \lim_{n \to \infty} \frac{1}{(n^2 - 1)^{1/2}} \cdot n = \lim_{n \to \infty} \frac{n}{(n^2 - 1)^{1/2}} = 1
\]

\[
\lim_{n \to \infty} \frac{n}{n^{1/2} + \text{junk}} = 0 < 1 < \infty
\]

\[
\sum b_n = \sum \frac{1}{n} \text{ diverges (harmonic series or p-test where p=1)}
\]

**A.S.T**

\[
\lim_{n \to \infty} |a_n| = 0 ; \lim_{n \to \infty} \frac{1}{\sqrt{n^2 - 1}} = \lim_{n \to \infty} \frac{1}{(n^2 - 1)^{1/2}} = \frac{1}{(\infty)^{1/2}} = 0
\]

\[
f(x) = |a_n| = f(x) \text{ is decreasing}
\]

\[
f(x) = (n^2 - 1)^{-1/2}
\]

\[
f'(x) = -\frac{1}{2} (n^2 - 1)^{-3/2} (2n) < 0
\]

**cond conv?**

\[
\exists L = \lim_{n \to \infty} f(x) = 0
\]

\[
\therefore \text{ the given series is not abs conv. (\Rightarrow diverges)}
\]

\[
\therefore \text{ the given series is cond convergent due to the A.S.T}
\]
11. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

\[ \sum_{n=8}^{\infty} (-1)^{n} \frac{n!}{(2n-1)!} \]

- Absolutely convergent
- Conditionally convergent
- Divergent

But before you get started ... let

\[ a_n = \frac{n!}{(2n-1)!} \]

Then

\[ a_{n+1} = \frac{(n+1)!}{(2n+1)!} \]

Next, simplify \( \frac{a_{n+1}}{a_n} \) so that it has NO factorial sign (that is a ! sign) in it.

\[ \frac{a_{n+1}}{a_n} = \frac{n+1}{(2n+1)(2n)} \]

Ok, now you should be ready to finish off the problem and check the correct box above.

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+1)(2n+2-1)!} \]

\[ = \frac{(n+1)(2n)!}{(2n+1)(2n)(2n+1)!} \]

\[ = \lim_{n \to \infty} \frac{n+1}{(2n+1)(2n)} \]

\[ = \lim_{n \to \infty} \frac{n^2 + 2n}{4n^2 + 2n} = \lim_{n \to \infty} \frac{1}{8n + 2} = 0 \]

\[ L = \lim_{n \to \infty} \frac{1}{8n + 2} = 0 \]

\[ L < 1 \text{ con} \]

\[ L > 1 \text{ div} \]

\[ \therefore \text{ The given series is absolutely convergent } \]

\[ \therefore \text{ The given series converges by the ratio test } \]
12. Consider the formal power series

\[ \sum_{n=1}^{\infty} \frac{5^n(x-3)^n}{n^2} \]

Hint: \[ \frac{5^{n+1}(x-3)^{n+1}}{(n+1)^2} - \frac{5^n(x-3)^n}{n^2} = \frac{5^{n+1}(x-3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{5^n(x-3)^n} = 5 \cdot \frac{|x-3|}{n+1} \cdot \left( \frac{n}{n+1} \right)^2 \]

The center is \( x_0 = \frac{3}{1} \) and the radius of convergence is \( R = \frac{1}{5} \).

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{5 \cdot \frac{n}{n+2}}{n^2 + 2n+1} = \frac{5n}{n^2 + 2n+1} \]

Apply the ratio test:

\[ |x-3| \lim_{n \to \infty} \frac{5n^2}{5n^2 + 2n+1} = 5 |x-3| < 1 \]

\[ |x-3| < \frac{1}{5} \]

\[ |x+3| < \frac{1}{5} \]

\[ \text{Center: } 3, \text{ Radius: } \frac{1}{5}, \text{ Convergent for } |x-3| < \frac{1}{5}, |x+3| < \frac{1}{5} \]

\[ x = \frac{14}{5}, x = \frac{14}{5} - \frac{2n}{3} \]

\[ a_n = 5^n, \text{ Center: } 3, \text{ Radius: } \frac{1}{5}, \text{ Convergent for } \frac{5}{2} < 1 \]

\[ \frac{5n^2}{5n^2 + 2n+1} = \frac{5}{2} \text{ not abs conv} \]

\[ \lim_{n \to \infty} \frac{5n^2}{2n+1} = \frac{5}{2} \text{ not abs conv} \]

\[ \lim_{n \to \infty} \frac{5n^2}{5^2n^2} = \frac{5}{5} = 1 \text{ not abs conv} \]

\[ \lim_{n \to \infty} \frac{5n^2}{2n+1} = \frac{5}{2} \text{ not abs conv} \]

\[ \lim_{n \to \infty} \frac{5n^2}{5^2n^2} = \frac{5}{2} \text{ not abs conv} \]
endpoints:

\[ \sum_{n=1}^{\infty} \frac{5^n \left( \frac{14}{5} - 3 \right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^n (-\frac{1}{5})^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^n (-1)^n \left( \frac{1}{5} \right)^n}{n^2} \]

abs conv? \[ \exists \lim_{n \to \infty} \frac{5^n \left( \frac{1}{5} \right)^n}{n^2} = \frac{1^{2n}}{n^2} = \frac{1}{n^2} \]

p-test

\[ p = 2 \]

\[ p > 1 \]

converges absolutely

\[ = \frac{16}{5} - \frac{15}{5} = \frac{1}{5} \]

\[ \sum_{n=1}^{\infty} \frac{5^n \left( \frac{16}{5} - 3 \right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^n \left( \frac{1}{5} \right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ \therefore \text{converges absolutely} \]

\[ \sum \frac{1}{n} \text{ diverges} \]

\[ \sum \frac{\ln n}{n} \text{ converges} \]
13. Consider the formal power series

\[ \sum_{n=1}^{\infty} \frac{n!}{n^3} (x-2)^n \]

Hint: do the same kind of calculation as done in the hint for the previous problem.

The center is \( c_0 = \frac{2}{4} \) and the radius of convergence is \( R = \frac{c_0}{2} \).

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.

\[ \text{Ratio Test: } a_{n+1} = \frac{(n+1)!}{(n+1)^3} \frac{(x-2)^{n+1}}{} \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \frac{(x-2)^{n+1}}{} \right| \]

\[ \lim_{n \to \infty} \frac{(n+1)!}{n!} (x-2) = \frac{n^3}{n!} \frac{(x-2)^n}{(x-2)^n} = |x-2| \lim_{n \to \infty} \frac{(n+1)!}{n!} \frac{(x-2)^n}{(x-2)^n} \]

\[ \lim_{n \to \infty} \frac{n^3}{n!} (x-2) = \frac{\infty}{\infty} = \infty \]

\[ |x-2| \lim_{n \to \infty} \frac{n^3}{n!} (x-2) = |x-2| \lim_{n \to \infty} \frac{n^3}{n!} (x-2) = \frac{\infty}{\infty} = \infty \]

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14. Consider the point, in polar coordinates,

\[ P = (r, \theta) = \left(2, \frac{\pi}{6}\right). \]

In cartesian coordinates, the point \( P \) is given by

\[ P = (x, y) = \left(\frac{\sqrt{3}}{2}, 1\right). \]

Below graph, and CLEARLY label, the following points.

- \( P = \left(2, \frac{\pi}{6}\right) \)
- \( Q = \left(-2, -\frac{\pi}{6}\right) \)
- \( R = \left(2, -\frac{\pi}{6}\right) \)
- \( S = \left(-2, -\frac{\pi}{6}\right) \)

\[ y = \frac{1}{2} \]

\[ y = 1 \]
15. Consider the curve in polar coordinate

\[ r = 3 \sin(2\theta) \]

15a. The period of \( r = 3 \sin(2\theta) \) is \( \frac{2\pi}{2} = \pi \).

15b. The period of \( r = 3 \sin(2\theta) \) is \( \frac{\pi}{4} \).

15c. Make a chart, as we did in class, to help you graph \( r = 3 \sin(2\theta) \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 2\theta )</th>
<th>( \sin(2\theta) )</th>
<th>( r = 3 \sin(2\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{\pi}{2} )</td>
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<td>( \frac{\pi}{2} )</td>
<td>( 3\pi )</td>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( 2\pi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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</tr>
<tr>
<td>( 2\pi )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15d. Graph \( r = 3 \sin(2\theta) \).
16. Express the area enclosed by \( r = 3 \sin(2\theta) \) as an integral with respect to \( \theta \)

(OK ... with respect to \( \theta \) means a \( d\theta \) in there).

(You do not have to evaluate this integral.)

\[
\text{area} = \frac{1}{2} \cdot 4 \int_{\theta=0}^{\pi/2} \left[ 3 \sin(2\theta) \right]^2 d\theta
\]

\[
A = \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[ r(\theta) \right]^2 d\theta \quad \text{where} \quad r = f(\theta)
\]

\[
A = \frac{1}{2} \cdot 4 \int_{\theta=0}^{\pi/2} \left[ 3\sin(2\theta) \right]^2 d\theta
\]

due to

symmetry