p.547,#8.2.6: Let \( c \) consist of straight lines joining \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\) and let \( S \) be the triangular surface with these vertices. Verify Stokes' Theorem directly with \( F = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \).

Solution: (The sneaky easy way.) Note that \( F = \nabla (xyz) \) so \( \text{curl} \ F = \nabla f \) so \( \text{curl} \ F \cdot dS = 0 \) so \( \int_S \text{curl} \ F \cdot dS = 0 \). \( c \) begins and ends at the same point, say \( p = (1, 0, 0) \). So \( \int_c F \cdot ds = \int_c \nabla f \cdot ds = f(p) - f(p) = 0 \). □

p.528,#8.1.2: Find the area of the disk \( D \) of radius \( R \) using Green’s Theorem.

Solution: Let \( \omega = -y^2 \, dx + x^2 \, dy \). Let \( \eta: D \rightarrow D \) be the identity mapping, which is definitely a parameterization. \( d\omega = dx \wedge dy \), so \( \int_{\partial \eta} \omega = \int_{D} (dx \wedge dy)(1) \, dx \, dy = \int_{D} dx \, dy \), which is the area of the disk \( D \). By Green’s Theorem this is equal to \( \int_{\partial \eta} \omega \). Since \( \zeta: (-\pi, \pi) \rightarrow \partial D: t \mapsto \langle R \cos t, R \sin t \rangle \) parameterizes the boundary curve of \( D \) and \( D\zeta(t) = \langle -R \sin t, R \cos t \rangle \), and \( n_{out} = \langle \cos t, \sin t \rangle \) is outward pointing at the point \( \zeta(t) \in \partial D \), we have

\[
\int_{\partial \eta} \omega = \text{sign} \det \begin{pmatrix} \cos t & -R \sin t \\ \sin t & R \cos t \end{pmatrix} \int_{\zeta} \omega \\
= \text{sign} R \int_{(-\pi, \pi)} \left( -\frac{1}{2} R \sin t \, dx + \frac{1}{2} R \cos t \, dy \right) \left( -\frac{R \sin t}{R \cos t} \right) \, dt \\
= \frac{R^2}{2} \int_{-\pi}^{\pi} \sin^2 t + \cos^2 t \, dt = \pi R^2.
\]

So the area of the disk \( D \) is \( \pi R^2 \). □

p.528,#8.1.4: a) Using the Divergence Theorem, show that \( \int_{\partial D} F \cdot n \, ds = 0 \) where \( F(x, y) = yi - xj \) and \( D \) is the unit disk. b) Verify this directly.

Solution: a) Divergence Theorem is the same as the Divergence version of Green’s Theorem. Let \( \eta, \zeta \) be as in problem #8.1.2 with \( R = 1 \). \( \int_{\partial D} F \cdot n \, ds \) is the classical notation for the flux of \( F \) through the curve \( \partial D \), i.e. for \( \int_{\zeta} \ast F^T \). We checked that the sign is +1 in #8.1.2 so Green’s Theorem says that \( \int_{\zeta} \ast F^T = \int_{\eta} d \ast F^T \). \( F^T = y \, dx - x \, dy \) so \( \ast F^T = y(\ast dx) - x(\ast dy) = y \, dy - x(- \, dx) = x \, dx + y \, dy \), since in 2D
\[ *dx = dy \text{ and } *dy = -dx. \] Therefore \( \int_\eta d \ast F^T = \int_\eta 0 = 0. \]

b) \( \int_\zeta \ast F^T = \int_{(-\pi,\pi)} \left( \cos t \; dx + \sin t \; dy \right) \left( -\sin t \over \cos t \right) dt = \int_{-\pi}^{\pi} (-\cos t \sin t + \sin t \cos t) dt = 0. \]

\( \square \)

p.528,\#8.1.10: Let \( D \) be a region for which Green’s Theorem holds. Suppose \( f \) is harmonic, i.e. \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \) on \( D \). Prove that \( \int_{\partial D} \frac{\partial f}{\partial y} \; dx - \frac{\partial f}{\partial x} \; dy = 0. \)

**Solution:** Let \( \omega = \frac{\partial f}{\partial y} \; dx - \frac{\partial f}{\partial x} \; dy \), \( \eta \) parameterize \( D \), \( \zeta \) parameterize \( \partial D \) so that the sign is +1. Then by Green’s Theorem we have

\[
\int_{\partial D} \frac{\partial f}{\partial y} \; dx - \frac{\partial f}{\partial x} \; dy = \int_\zeta \omega = \int_\eta d\omega = \int_\eta \frac{\partial^2 f}{\partial y^2} \; dy \wedge dx - \frac{\partial^2 f}{\partial x^2} \; dx \wedge dy
\]

\[
= \int_\eta (-\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}) \; dx \wedge dy = \int_\eta (-0) \; dx \wedge dy = 0. \]

\( \square \)

p.529,\#8.1.12: Let \( P(x,y) = -\frac{y}{x^2+y^2} \) and \( Q(x,y) = \frac{x}{x^2+y^2} \). Assuming \( D \) is the unit disk, investigate why Green’s Theorem fails for this \( P \) and \( Q \).

**Solution:** They mean Green’s Theorem for the form \( \omega = P \; dx + Q \; dy \). This needs to be a 1-form on some open subset \( U \) of \( \mathbb{R}^2 \) containing the closed unit disk. In order to be a 1-form it needs to be at least \( C^1 \) as a function from \( U \) into \( \Lambda^1(\mathbb{R}^2) \). If it were \( C^0 \) it would be bounded near \( (x,y) = (0,0) \), but it is not. For example \( P(0,y) = -1/y \) and \( Q(x,0) = 1/x \). Thus \( \omega \) is not even continuous near \( (0,0) \), and hence does not satisfy the hypotheses of Green’s Theorem. So we should not be surprised if the conclusion of Green’s Theorem fails to hold. This answers the why question, without in fact showing that the conclusion of Green’s Theorem fails to hold. \( \square \)