The lecture covered the completeness of the real number line. Given a subset \( A \subset \mathbb{R} \), we say \( A \) is bounded above (or bounded below, respectively) if there exists a real number \( x \) such that for every \( y \in A \), \( y \leq x \) (respectively, \( x \leq y \)).

**Axiom** (Completeness axiom). If \( C \subset \mathbb{R} \) is non-empty and bounded above, then \( C \) has a least upper bound. If \( C \subset \mathbb{R} \) is non-empty and bounded below, then \( C \) has a greatest lower bound.

Note that if a set has an upper bound, then it has many of them. Indeed, if \( x \) is an upper bound for the set \( A \), then every number larger than \( x \) is also an upper bound. However, there can be only one least upper bound for a set. This follows from the fact that for real numbers \( r, s \), if \( s \leq r \leq s \), then \( r = s \). We refer to the least upper bound of the set \( A \) as the supremum of \( A \), denoted \( \sup A \). For convenience, we define \( \sup \emptyset = -\infty \) and \( \sup A = \infty \) if \( A \) is not bounded above. We refer to the greatest lower bound (again, there is only one) of the set \( A \) as the infimum of \( A \), denoted \( \inf A \). For convenience, we define \( \inf \emptyset = \infty \) and \( \inf A = -\infty \) if \( A \) is not bounded below.

Intuitively, this says that the real number line has no holes in it. To see the significance of the axiom above, let us see how it fails if we try to replace \( \mathbb{R} \) with \( \mathbb{Q} \). If \( A \) is the set of rational numbers less than \( \pi \), then \( A \) is bounded above (every rational number exceeding \( \pi \) is an upper bound). However, there is not a least rational number which is an upper bound for the set \( A \), since for any \( \pi < q \), there exists a rational number between \( \pi \) and \( q \).

We also have another axiom describing this completeness property of the real line.

**Axiom** (Dedekind cut axiom). Suppose \( A, B \subset \mathbb{R} \) are subsets of the real line such that

\[
(i) \ A \cup B = \mathbb{R}, \\
(ii) \ A \cap B = \emptyset, \\
(iii) \ A, B \neq \emptyset, \\
(iv) \text{for any } \ x \in A \text{ and } y \in B, \ x < y.
\]

Then there exists a real number \( r \) such that for any \( x \in A \) and any \( y \in B \), \( x \leq r \leq y \).

The content of the axiom is that if we break the real line into two subsets so that one lies entirely to the left of the other, there has to be some real number which is the boundary between \( A \) and \( B \).

**Theorem 1.1.** The completeness axiom is equivalent to the Dedekind cut axiom.

**Proof.** Assume the completeness axiom holds. Assume \( A, B \) are as in the statement of the Dedekind cut axiom. Note that \( A \) is non-empty by (iii). Note also that \( A \) is bounded above, since by (iii), there exists \( y \in B \) and by (iv), \( y \) is an upper bound for \( A \). By the completeness axiom, there exists a least upper bound \( r = \sup A \) of \( A \). We claim that this \( r \) satisfies the conclusion of the Dedekind cut axiom. By definition, since \( r \) is the least upper bound for \( A \),
and therefore is an upper bound for $A$, $x \leq r$ for every $x \in A$. For any $y \in B$, $r \leq y$, since, as we mentioned above, $y$ is an upper bound for $A$ and $r$ is the least upper bound for $A$.

Next, assume the Dedekind cut axiom holds and suppose $C$ is non-empty and bounded above. Let $B$ consist of all upper bounds for $C$ and let $A = \mathbb{R} \setminus B$. In order to apply the Dedekind cut axiom to the sets $A, B$, we need to check that the four conditions are satisfied. Of course, $A \cup B = \mathbb{R}$ and $A \cap B = \emptyset$. Since $C$ is bounded above, $C$ has an upper bound, and $B \neq \emptyset$. Indeed, if $x \in C$, then $x - 1$ is not an upper bound for $C$ (since $x - 1 < x$), and so $x - 1 \in A$. Last, if $x \in A$ and $y \in B$, then $x < y$. If it were not so, then $y \leq x$, and since $y$ is an upper bound for $A$, so is $x$. But this would mean $x \in B$ and $x \notin A$, a contradiction. Thus the four conditions of the Dedekind cut axiom are satisfied, and we may apply it to these choices of $A, B$. Let $r$ be such that for every $x \in A$ and $y \in B$, $x \leq r \leq y$. We claim that $r$ is the least upper bound of $C$. First, we need to show that $r$ is an upper bound for $C$. If it is not, then there exists $x \in C$ such that $r < x$. Then if $s = \frac{r + x}{2}$, $r < s < x$. Then $s$ is also not an upper bound for $C$, since $s < x$, so $s \in A$. However, $r < s$, contradicting our choice of $r$. Therefore the assumption that $r$ is not an upper bound for $C$ least to a contradiction, and it must be that $r$ is an upper bound for $C$. We next need to show that it is the least upper bound for $C$. If it were not, there would exist $y < r$ which is also an upper bound for $C$. But this means $y \in B$, and $y < r$ contradicts our choice of $r$. Thus $r$ is the least upper bound for $C$, and $C$ has a least upper bound.

In order to see that any non-empty, bounded below set has a greatest lower bound, we may argue similarly to the previous paragraph. We begin by letting $A$ be the set of all lower bounds for $C$ and $B = \mathbb{R} \setminus A$. We leave it to the student to modify the previous paragraph to prove this case.

We remark that we may offer another proof of the second claim of the completeness axiom using the first part. This is because $C$ is non-empty and bounded below if and only if $-C := \{-x : x \in C\}$ is non-empty and bounded above. If $r$ is the supremum of $-C$, then $-r$ is the greatest lower bound of $C$.

2. Thursday, 14 January

2.1. Mathematical induction. In this lecture, we briefly discussed proof by induction. Suppose that for some natural number $k$, we have propositions $P_k, P_{k+1}, P_{k+2}, \ldots$. That is, we have a sequence of assertions that we wish to prove. Then we may prove all of the statements by performing the two following steps:

(i) Prove that the first statement $P_k$ is true (the base step).
(ii) Prove that if $P_{n-1}$ is true for some $n > k$, then $P_n$ is also true.

This process is often compared to dominoes: In order to prove all of the claims, you need to knock down the first domino (the base step) and prove that knocking down one domino will knock down the next (the inductive step).
Example 2.1. For every real number \( r \neq 0,1 \) and every \( n = 0,1,\ldots \),

\[
\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.
\]

Recall that \( \sum_{i=0}^{n} r^i = 1 + r + \ldots + r^n \). We prove this statement by induction on \( n \). The base step is proving the result is true when \( n = 0 \). But the left side is \( \sum_{i=0}^{0} r^i = r^0 = 1 \) and the right side is \( \frac{r^{0+1} - 1}{r - 1} = \frac{1}{r - 1} = 1 \), and so the base step is true.

Next, assume the statement is true for \( n - 1 \). That is, assume \( \sum_{i=0}^{n-1} r^i = \frac{r^{n-1} - 1}{r - 1} \). Then

\[
\sum_{i=0}^{n} r^i = r^0 + r^1 + \ldots + r^{n-1} + r^n = \left( \sum_{i=0}^{n-1} r^i \right) + r^n
\]

\[
= \frac{r^n - 1}{r - 1} + r^n = \frac{r^n - 1 + r^{n+1} - r^n}{r - 1} = \frac{r^{n+1} - 1}{r - 1}.
\]

This gives the inductive step.

We cannot help but offer another short proof of the claim above. Note that it is sufficient to prove that

\[
(1 + r + r^2 + \ldots + r^n)(r - 1) = r^{n+1} - 1.
\]

But if we multiply the left side out, we obtain

\[
(1 + r + r^2 + \ldots + r^n)r + (1 + r + r^2 + \ldots + r^n)(-1) = r + r^2 + \ldots + r^n + r^{n+1} - 1 - r - r^2 - \ldots - r^n
\]

\[
= r^{n+1} - 1,
\]

since we have cancellation of all the terms except the rightmost term on the top line and the leftmost term on the second line.

The method of induction above is sometimes called weak induction. There is a stronger form of induction, which involves the following steps:

(i) Prove that \( P_k \) is true (the same base step).

(ii) Prove that if \( P_i \) is true for every \( k \leq i < n \), then \( P_n \) is true.

This is stronger than weak induction mentioned above, because the inductive step has to assume that every step before the \( n \)th step is true in order to prove the \( n \)th step, whereas the weak version only has to assume that the \( n-1 \)st step is true in order to prove the \( n \)th step.

2.2. Open and closed sets, DeMorgan’s laws.

Proposition 2.1. Let \( X \) be any set and let \( (U_i)_{i \in I} \) be a collection of subsets of \( X \). For any subset \( S \) of \( X \), we let \( S^C \) denote the complement (in \( X \)) of \( S \). Then

\[
\left( \bigcup_{i \in I} U_i \right)^C = \bigcap_{i \in I} U_i^C \quad \text{and} \quad \left( \bigcap_{i \in I} U_i \right)^C = \bigcup_{i \in I} U_i^C.
\]
Sketch. The first two sets are both precisely those members of $X$ which do not lie in any of the $U_i$ sets. The last two sets are both precisely those members of $X$ which fail to be in at least one of the $U_i$ sets.

We may have also heard of DeMorgan’s laws in logic, which addresses the behavior of “and” (which takes the place of unions above) and “or” (which takes the place of intersections above) under negation (denoted $\neg$, which takes the place of complementation). Recall that $\bigvee_{i \in I} P_i$ means “at least one of $P_i$” and $\bigwedge_{i \in I} P_i$ means “all $P_i$.” If $I$ consists of two propositions, say $P$ and $Q$, then $P \lor Q$ means “$P$ or $Q$,” and $P \land Q$ means ”$P$ and $Q.” If $(P_i)_{i \in I}$ is a collection of propositions, $\neg \bigvee_{i \in I} P_i = \bigwedge_{i \in I} \neg P_i$ and $\neg \bigwedge_{i \in I} P_i = \bigvee_{i \in I} \neg P_i$. The sketch of the proof is as above: The first two propositions can be thought of as “each $P_i$ is false” and the last two as “at least one $P_i$ is false.”

We say that a subset $U$ of $\mathbb{R}$ is open if for every $x \in U$, there exists $r > 0$ such that $(x-r, x+r) \subset U$. Intuitively, we think of this as meaning that none of the points of $U$ lie on the “boundary” of $U$ (note that we have not defined “boundary” at this point, but we will later give a formal definition of the term). Let us give some examples.

Claim. The following are open sets.

(i) $\emptyset$,
(ii) $\mathbb{R}$,
(iii) $(a, b)$ for any real numbers $a < b$,
(iv) $\bigcup_{i \in I} U_i$ if each $U_i$ is open,
(v) $(-\infty, b)$ for any real number $b$,
(vi) $(a, \infty)$ for any real number $a$,
(vii) $\bigcap_{i=1}^n U_i$ if each $U_i$ is open.

The following are closed sets.

(i) $\mathbb{R}$,
(ii) $\emptyset$,
(iii) $[a, b]$ for any real numbers $a \leq b$,
(iv) $\bigcap_{i \in I} C_i$ if each $C_i$ is closed,
(v) $(-\infty, a]$ for any real number $a$,
(vi) $[b, \infty)$ for any real number $b$,
(vii) $\bigcup_{i=1}^n C_i$ if each $C_i$ is closed.

Proof. We prove the indicated sets are open.

(i) The empty set is vacuously open, since the condition in the definition of open cannot fail for any member of the open set (since there are no members for which it could fail).

(ii) The set $\mathbb{R}$ is open, since for any $x \in \mathbb{R}$, we may take any positive $r > 0$ and $(x-r, x+r) \subset \mathbb{R}$.

(iii) For any $a < b$ and $x \in (a, b)$, Let $r = \min\{b-x, x-a\} > 0$. Then $(x-r, x+r) \subset (a, b)$. 

(iv) Suppose that \( x \in \bigcup_{i \in I} U_i \). Then there exists \( j \in I \) such that \( x \in U_j \). Since \( U_j \) is open, there exists \( r > 0 \) such that \( (x - r, x + r) \subset U_j \subset \bigcup_{i \in I} U_i \).

(v) \( (-\infty, b) = \bigcup_{a < b} (a, b) \).

(vi) \( (a, \infty) = \bigcup_{a < b} (a, b) \).

(vii) Suppose that \( x \in \bigcap_{i=1}^n U_i \). Then for each \( 1 \leq i \leq n \), \( x \in U_i \). Since \( U_i \) is open, there exists \( r_i > 0 \) such that \( (x - r_i, x + r_i) \subset U_i \). Then if \( r_i = \min\{r_1, \ldots, r_n\} \), \( (x - r, x + r) \subset \bigcap_{i=1}^n (x - r_i, x + r_i) \subset \bigcap_{i=1}^n U_i \).

We next prove the sets are closed.

(i) The complement of \( \mathbb{R} \) is \( \emptyset \), which is open, so \( \emptyset \) is closed.

(ii) The complement of \( \emptyset \) is \( \mathbb{R} \), which is open, so \( \emptyset \) is closed.

(iii) The complement of \([a, b] = (-\infty, a) \cup (b, \infty)\), which is a union of open sets, and is therefore open. Thus \([a, b]\) is closed.

(iv) By DeMorgan’s Law, the complement of \( \bigcap_{i \in I} C_i = \bigcup_{i \in I} \mathbb{R} \setminus C_i \). This is a union of open sets, and is therefore open. Each set \( \mathbb{R} \setminus C_i \) is open since \( C_i \) is closed.

(v) \( (-\infty, a] \) is the complement of the open set \((a, \infty)\), and is therefore closed.

(vi) \([b, \infty) \) is the complement of the closed set \((-\infty, b)\), and is therefore open.

(vii) By DeMorgan’s Law, the complement of \( \bigcup_{i=1}^n C_i \) is \( \bigcap_{i=1}^n \mathbb{R} \setminus C_i \), which is a finite intersection of open sets, and is therefore open.

\( \square \)

Example 2.2. Above, we showed that the intersection of finitely many open sets is open. It is not true that the intersection of infinitely many open sets is open. Indeed,

\[
\bigcap_{x>0} (-\infty, x) = (-\infty, 0],
\]

which is not open. This is because for any \( r > 0 \), \( (-r, r) = (0 - r, 0 + r) \notin (-\infty, 0] \).

Taking complements yields an example of an infinite union of closed sets which is not closed:

\[
\bigcup_{x>0} [x, \infty) = (0, \infty).
\]

Example 2.3. There are many sets which are neither open nor closed. A simple example of such a set is \([0, 1)\). This set is not open, because there is no open interval around 0 which stays inside this set. This set is also not closed, since the complement is \((-\infty, 0) \cup [1, \infty)\), and there is no open interval around 1 which stays inside this set.