Applications of Integration I: Areas Between Curves

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Overview
This lab will help to develop your understanding of the definite integral as area for non-negative functions. In this lab you will explore how to use Maple to visualize a given region, determine points of intersection if necessary, and determine its area. You will discover that for some regions it is best to regard \( x \) as a function of \( y \).

Maple Essentials
New Maple commands introduced in this lab include:

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
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<tbody>
<tr>
<td><code>fsolve</code></td>
<td>numerically solves one or more equations for their unknowns</td>
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<tr>
<td>( \text{fsolve}(x^2-4=0); ) returns the values -2.0, 2.0</td>
<td></td>
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<tr>
<td>( \text{fsolve}(x^2-4=0, x=-3..-1); ) returns only -2.0</td>
<td></td>
</tr>
<tr>
<td><code>int(f(x), x=a..b);</code></td>
<td>evaluates ( \int_{a}^{b} f(x),dx )</td>
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The Area of a Region by Slicing maplet is available from the course website:

http://www.math.sc.edu/calclab/142L-S14/labs Area of a Region by Slicing

This maplet demonstrates finding the area of a region by slicing. The student may generate a random example or modify or make their own problem.

Preparation
In the case where both \( f(x) \) and \( g(x) \) are positive, it is very easy to see that if we want to find the area between \( f(x) \geq g(x) \) (both continuous on \([a, b]\)) we would do the following:

\[
A = \int_{a}^{b} f(x)\,dx - \int_{a}^{b} g(x)\,dx = \int_{a}^{b} (f(x) - g(x))\,dx.
\]

In general, for a given region with top curve \( y_T \) and bottom curve \( y_B \) we can draw a typical approximating rectangle with height \( (y_T - y_B) \) and width \( \Delta x \) and calculate the area as follows:

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\Delta x = \int_{a}^{b} (y_T - y_B)\,dx
\]

Sometimes regions are best treated by regarding \( x \) as a function of \( y \). If a region is bounded by curves with equations \( x = f(y) \), \( x = g(y) \), \( y = c \), and \( y = d \), where \( f \) and \( g \) are continuous and \( f(y) \geq g(y) \) for \( c \leq y \leq d \) then its area is

\[
A = \int_{c}^{d} (f(y) - g(y))\,dy.
\]

In general, for a given region with right boundary \( x_R \) and left boundary \( x_L \) we can draw a typical approximating rectangle with height \( \Delta y \) and width \( (x_R - x_L) \) and calculate the area as follows:

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} (x_R - x_L)\Delta y = \int_{c}^{d} (x_R - x_L)\,dy
\]
Related Course Material
Section 6.1 in Stewart. Section 7.1 in CalcLabs.

Activity 1
Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.
- We will first assign our functions as $f(x)$ and $g(x)$.
  > f:=x-> x^2;
  > g:=x-> x+6;
- Next, we plot the functions to get an idea of the region we are considering.
  > plot([f(x),g(x)], x=-5..5, y=-10..10, color=[red, blue]);
- We find the intersection points using fsolve.
  > fsolve(f(x)=g(x));
- We then evaluate the integral to find the area. Notice for this example $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  > Area:=int(g(x)-f(x), x=-2..3);

Activity 2
Find the area of the region enclosed between $y = -0.128x^3 + 1.728x^2 - 5.376x + 2.864$ and $y = \ln x$.
- We will first assign our functions as $f(x)$ and $g(x)$.
  > f:=x-> -0.128*x^3+1.728*x^2-5.376*x+2.864;
  > g:=x-> ln(x);
- Next, we plot the functions to get an idea of the region we are considering. Be careful with the window this time. There are two separate areas enclosed by the functions.
  > plot([f(x),g(x)], x=0..10, y=-10..10, color=[red, blue]);
- We can see from the graph that there are three points of intersection, but fsolve (without a specified interval) will only give one, so we will write a separate command to find and assign each intersection point.
  > a:= fsolve(f(x)=g(x), x=0..2);
  > b:= fsolve(f(x)=g(x), x=2..6);
  > c:= fsolve(f(x)=g(x), x=8..10);
- For our first area, notice that between $a$ and $b$, $g(x)$ is the top curve and $f(x)$ is the bottom curve.
  > A1:=int(g(x)-f(x), x=a..b);
- For our second area, notice that between $b$ and $c$, $f(x)$ is the top curve and $g(x)$ is the bottom curve.
  > A2:=int(f(x)-g(x), x=b..c);
- We find our total area by adding $A1$ and $A2$.
  > Area:= A1 + A2;

Activity 3
Find the area of the region enclosed between $x = y^2$ and $x = y + 2$.
- We will first assign our expressions as $f(y)$ and $g(y)$.
  > f:=y-> y^2;
  > g:=y-> y+2;
- We can use the fsolve command to find the points of intersection.
  > f(y)=g(y);
- Once you have the points of intersection, you can use the Area of a Region by Slicing maplet to get a clear picture of the region.
- We then evaluate the integral to find the area. Notice for this example $g(y)$ is the right boundary and $f(y)$ is the left boundary. Also, we are integrating with respect to $y$.
  > Area:=int(g(y)-f(y), y=-1..2);

Assignment
With the help of Maple, work out the problems assigned by your lab instructor. Clearly identify your answers on your Maple worksheet. Make sure you answer each question completely.