Exam 3 will be based on:

- Sections 8.1 - 8.4;
  At minimum, you need to understand how to do the homework problems.

Topic List (not necessarily comprehensive):

You will need to know: theorems, results, and definitions from class.

§8.1: Green’s Theorem.

**Theorem.** Let $D \subseteq \mathbb{R}^2$ be a region enclosed by a simple closed curve $C$. Suppose that $P$, $Q : D \to \mathbb{R}$ are $C^1$. Then we have

$$
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
$$

Notes:

- **Application: Area.** Let $D \subseteq \mathbb{R}^2$ be enclosed by a simple closed curve $C$. Then we have
  $$
  \text{Area}(D) = \frac{1}{2} \int_C x \, dy - y \, dx.
  $$
- **Alternative formulations.**
  - Let $\vec{F} = P\vec{i} + Q\vec{j}$. Then we have
    $$
    \int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} \, dA.
    $$
    This form looks like Stokes’ Theorem.
  - We have
    $$
    \int_{\partial D} \vec{F} \cdot \vec{n} \, ds = \iint_D \nabla \cdot \vec{F} \, dA.
    $$
    This is the “Divergence Theorem in the plane”. 
§8.2: Stokes’ Theorem.

**Theorem.** Suppose that $S$ is an oriented surface, suppose that $D \subseteq \mathbb{R}^2$ is a region to which Green’s Theorem applies, and suppose that $\Phi : D \rightarrow S$ is a $1-1$ parametrization of $S$. Let $\vec{F} : S \rightarrow \mathbb{R}^3$ be $C^1$. Then we have

\[ \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}. \]

**Notes:**
- The curve $\partial S$ has orientation induced by $S$: one traverses the curve $\partial S$ in the positive direction when the positive side of $S$ is on the left.
- On specializing $D = S$ and $\vec{F} = P\hat{i} + Q\hat{j}$, one obtains Green’s Theorem.

**Definition.** Let $C$ be a simple, oriented, closed curve, and let $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then the **circulation** of $\vec{F}$ around $C$ is $\int_C \vec{F} \cdot d\vec{s}$.

**Theorem.** Let $S \subseteq \mathbb{R}^3$ be a surface, let $P \in S$, and suppose that $S$ has orientation determined by a unit normal $\vec{n}$ to $S$. Then we have

\[ \lim_{A(S) \rightarrow 0} \frac{1}{A(S)} \int_{\partial S} \vec{F} \cdot d\vec{s} = \left[ (\nabla \times \vec{F}) \cdot \vec{n} \right] |_P = \text{curl } \vec{F} \cdot \vec{n} |_P, \]

where $A(S) = \text{Area}(S) \rightarrow 0$ means that $S$ shrinks smoothly and continuously to the point $P$.

**Notes:**
- The right side of the theorem gives the component of curl $\vec{F}$ normal to the surface $S$ at the point $P \in S$. The right side is maximized when $\vec{n}$ points in the same direction as $\nabla \times \vec{F}$ at $P$. I.e., curl $\vec{F}$ points in the direction of the axis about which $\vec{F}$ “rotates the most”.
- The left side of the theorem gives the circulation of $\vec{F}$ around the boundary of $S$ per unit area.

§8.3: Conservative vector fields.

**Definitions.** Let $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a $C^1$ vector field.

- The field $\vec{F}$ is **conservative** or a gradient field if and only if there exists $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ with $\nabla f = \vec{F}$. The function $f$ is a scalar potential for $\vec{F}$.
- The field $\vec{F}$ is **irrotational** if and only if $\nabla \times \vec{F} = \vec{0}$.
- The field $\vec{F}$ is incompressible or solenoidal if and only if $\nabla \cdot \vec{F} = 0$.
- Suppose that $\vec{F} = \nabla \times \vec{G}$. Then $\vec{G}$ is a vector potential for $\vec{F}$. 

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**Theorem.** Suppose that \( \vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is \( C^1 \) except at finitely many points. Then the following are equivalent.

1. The circulation of \( \vec{F} \) around the boundary of every simply closed curve \( C \) is zero:
   \[
   \int_C \vec{F} \cdot d\vec{s} = 0.
   \]
2. Line integrals along oriented simple curves are independent of path: for all oriented simple curves \( C_1, C_2 \) connecting points \( P \) and \( Q \), we have
   \[
   \int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}.
   \]
3. The vector field \( \vec{F} \) is conservative.
4. The vector field \( \vec{F} \) is irrotational.

**Notes:**

- The theorem continues to hold for 2-dimensional fields \( \vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). However, \( \vec{F} \) must be \( C^1 \) everywhere.
- To verify whether or not a vector field \( \vec{F} \) is conservative, one typically checks whether or not \( \nabla \times \vec{F} = \vec{0} \). (This is \( (3) \iff (4) \)).
- If \( \vec{F} \) is conservative, then one finds a potential function \( f \) via one of the following methods.
  1. We compute
     \[
     f = \int \frac{\partial f}{\partial x} \, dx + g_1(y, z), \quad f = \int \frac{\partial f}{\partial y} \, dy + g_2(x, z), \quad f = \int \frac{\partial f}{\partial z} \, dz + g_3(x, y).
     \]
     Since \( \vec{F} \) is conservative, one can find \( g_1, g_2, \) and \( g_3 \) for which the three expressions for \( f \) agree.
  2. We compute
     \[
     f(x, y, z) = \int_0^x F_1(t, 0, 0) \, dt + \int_0^y F_2(x, t, 0) \, dt + \int_0^z F_3(x, y, t) \, dt.
     \]
- The scalar potential \( f \) is unique up to addition by a constant \( C \).
Theorem. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$. Then $\vec{F}$ is incompressible (i.e., $\nabla \cdot \vec{F} = 0$) if and only if there exists $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\vec{F} = \nabla \times \vec{G}$ ($\vec{G}$ is a vector potential for $\vec{F}$).

Notes:

- If $\vec{F}$ is incompressible, one finds a vector potential $\vec{G}$ as follows. Compute
  
  $$G_1 = \int_0^z F_2(x, y, t) \, dt - \int_0^y F_3(x, t, 0) \, dt, \quad G_2 = -\int_0^z F_1(x, y, t) \, dt, \quad G_3 = 0.$$  

  Then $\nabla \times \vec{G} = \vec{F}$.
- A vector potential $\vec{G}$ is unique up to addition by a gradient field.

§8.4: The divergence theorem.

A surface $S$ is closed if and only if $\partial S = \emptyset$.

Theorem. Suppose that

- $W$ is an elementary symmetric region in $\mathbb{R}^3$.
- $\partial W$ is a closed, oriented surface bounding $W$.
- $\vec{F} : W \to \mathbb{R}^3$ is a $C^1$ vector field.

Then we have

$$\iiint_W \nabla \cdot \vec{F} \, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}.$$  

Note: The Divergence Theorem applies to regions which arise as finite unions of elementary symmetric regions.

Definition. Let $S \subseteq \mathbb{R}^3$ be a closed surface with outward unit normal $\vec{n}$, and let $\vec{F} : S \to \mathbb{R}^3$ be a $C^1$ vector field. Then the flux of $\vec{F}$ through $S$ is the mass of particles forced through $S$ by $\vec{F}$ per unit time. The rate of net outward flux of $\vec{F}$ through $S$ is $\iint_S \vec{F} \cdot \vec{n} \, dS$.

Fact. Let $W \subseteq \mathbb{R}^3$ be an elementary region, and let $P \in W$. Then we have

$$\lim_{V(W) \to 0} \frac{1}{V(W)} \iint_{\partial W} \vec{F} \cdot d\vec{S} = \left( \nabla \cdot \vec{F} \right)|_P = \text{div} \vec{F}|_P,$$

where $V(W) = \text{vol}(W) \to 0$ means that $W$ shrinks smoothly and continuously to $P$.

Note: The fact says that the divergence of $\vec{F}$ at a point $P \in W$ is the rate of net outflux of $\vec{F}$ per unit volume.