Math 550, Exam 2.  3/17/11. Name: __________________________

- Read problems carefully. Show all work.
- No notes, calculator, or text.
- The exam is approximately 15 percent of the total grade.
- There are 100 points total. Partial credit may be given.
- Write your full name in the upper right corner of page 1.
- Number the pages in the upper right corner.
- Do problem 1 on page 1, problem 2 on page 2, etc.
- Circle or otherwise clearly identify your final answer.

1. **(20 points):** Suppose that \( \vec{F}(x, y) = -y\vec{i} + 2x\vec{j} \), and that \( C \) is the triangle in the first quadrant with vertices \((0, 0)\), \((2, 0)\), and \((0, 4)\). Compute the work done by \( \vec{F} \) in moving a particle counter-clockwise around \( C \): \( \int_C \vec{F} \cdot d\vec{s} \).

(One method requires a computation of three line integrals, one for each side of \( C \).)

**Solution:** Let \( C_1, C_2 \), and \( C_3 \) be the base, right side, and left side of \( C \), respectively. We have parametrizations:

\[
C_1 : \vec{c}_1(t) = (0, 0) + t \cdot [(2, 0) - (0, 0)] = (2t, 0), \quad t \in [0, 1];
\]
\[
C_2 : \vec{c}_2(t) = (2, 0) + t \cdot [(0, 4) - (2, 0)] = (2, 0) + t(2, 4) = (2 - 2t, 4t), \quad t \in [0, 1];
\]
\[
C_3 : \vec{c}_3(t) = (0, 4) + t \cdot [(0, 0) - (0, 4)] = (0, 4 - 4t), \quad t \in [0, 1].
\]

We have

\[
\int_{C_1} \vec{F} \cdot d\vec{s} = \int_0^1 (4t\vec{j}) \cdot (2\vec{i}) \, dt = 0;
\]
\[
\int_{C_2} \vec{F} \cdot d\vec{s} = \int_0^1 (-4\vec{i} + 2(2 - 2t)\vec{j}) \cdot (-2\vec{i} + 4\vec{j}) \, dt = \int_0^1 (8t + 8(2 - 2t)) \, dt
\]
\[
= \int_0^1 (16 - 8t) \, dt = 16 - 4 = 12;
\]
\[
\int_{C_3} \vec{F} \cdot d\vec{s} = \int_0^1 (-4 - 4t)\vec{i} \cdot (-4\vec{j}) \, dt = 0.
\]

It follows that \( \int_C \vec{F} \cdot d\vec{s} = 12 \). Alternatively, if \( D \) is the region enclosed by \( C \), then Green’s Theorem gives

\[
\int_C \vec{F} \cdot d\vec{s} = \int_D (-y \, dx + 2x \, dy) = \int_D (2 - (-1)) \, dx \, dy = 3 \cdot \text{area}(D) = (3)(2)(4)/(2) = 12.
\]
2. **(15 points):** Let $C$ be an oriented curve in $\mathbb{R}^3$ parametrized by
\[
\vec{c}(t) = 5(1 - t^2)e^t \vec{i} + t^3 \vec{j} + \left(\frac{t + 1}{t^2 + 1}\right) \vec{k}, \quad 0 \leq t \leq 1.
\]
Let $f(x, y, z) = (z + 1)e^{xz^2 + y} + 5(y^3 + z)$. Compute $\int_C \nabla f \cdot d\vec{s}$.

(Do not attempt to compute this directly. What are the initial and terminal points of $C$?)

**Solution:** The initial point of $C$ is $\vec{c}(0) = (5, 0, 1)$; the terminal point is $\vec{c}(0) = (0, 1, 1)$. Therefore, we have
\[
\int_C \nabla f \cdot d\vec{s} = f(5, 0, 1) - f(0, 1, 1) = (2e^5 + 5) - (2e + 10) = 2e^5 - 2e - 5.
\]

3. **(20 points):** Let $D = [-1, 1] \times [0, 2] \subset \mathbb{R}^2$, and let $S \subset \mathbb{R}^3$ be a surface with parametrization $\Phi : D \to S$ given by $\Phi(u, v) = (u^2 + v, 2uv^2, u)$.

(a) Find a vector normal to $S$. At what points is the parametrization regular (smooth)?

**Solution:** We compute
\[
\vec{T}_u = 2u \vec{i} + 2v^2 \vec{j} + \vec{k};
\]
\[
\vec{T}_v = \vec{i} + 4uv \vec{j}.
\]

It follows that
\[
\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 2v^2 & 1 \\ 1 & 4uv & 0 \end{vmatrix} = \vec{i}(-4uv) - \vec{j}(1) + \vec{k}(8u^2v - 2v^2) = -4uv\vec{i} + \vec{j} + (8u^2v - 2v^2)\vec{k}.
\]

Since the $\vec{j}$ component is non-zero, the parametrization is everywhere regular.

(b) Write down a double integral giving the surface area of $S$.

**Do not attempt to evaluate it.**

**Solution:** We have $||\vec{T}_u \times \vec{T}_v|| = \left((16u^2v^2 + (8u^2v - 2v^2)^2 + 1)^{1/2}ight.$. Hence, the area is
\[
\text{area}(S) = \int_D \left((16u^2v^2 + (8u^2v - 2v^2)^2 + 1)^{1/2} \right) du dv.
\]

4. **(20 points):** Let $S \subset \mathbb{R}^3$ be the surface bounded by the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 4)$. 

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(a) Compute a vector normal to $S$.

**Solution:** It suffices to compute:

\[
\vec{N} = [(1, 0, 0) - (0, 2, 0)] \times [(1, 0, 0) - (0, 0, 4)] = (1, -2, 0) \times (1, 0, -4) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & -2 & 0 \\
1 & 0 & -4
\end{vmatrix}
\]

\[
= 8\vec{i} + 4\vec{j} + 2\vec{k}.
\]

(b) Use the normal vector to give the plane determined by $S$.

**Solution:** The plane has equation \( \vec{N} \cdot (\vec{x} - 1)\vec{i} + y\vec{j} + z\vec{k} = 8(\vec{x} - 1) + 4y + 2z = 0 \), which gives \( 4x - 4 + 2y + z = 0 \), or \( z = 4 - 4x - 2y \).

(c) Compute the surface integral \( \iint_S (z + 4x + 2y) \, dS \).

**Solution:** From part (a), we see that a unit vector normal to the surface is \( \vec{n} = \frac{\vec{N}}{||\vec{N}||} \). Hence, we have \( \vec{n} \cdot \vec{k} = \frac{2}{\sqrt{64+16+4}} = \frac{1}{\sqrt{21}} \). It follows that \( dS = \sqrt{21} \, dx \, dy \). Therefore, we compute

\[
\sqrt{21} \iint_D (4 - 4x - 2y + 4x + 2y) \, dy \, dx = 4\sqrt{21} \iint_D dy \, dx = 4\sqrt{21} \cdot \text{area}(D)
\]

\[
= (4)(\sqrt{21})(2)(4)/2 = 16\sqrt{21}.
\]

5. **(25 points):** Let \( \vec{F} = x\vec{i} - y\vec{j} + z\vec{k} \). Let \( S = S_{\text{sides}} \cup S_{\text{right cap}} \cup S_{\text{left cap}} \) be an open cylinder tipped on its side with

\[
S_{\text{sides}} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 9; 1 \leq y \leq 3\};
\]

\[
S_{\text{right cap}} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 < 9; y = 3\};
\]

\[
S_{\text{left cap}} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 < 9; y = 1\}.
\]

Suppose also that \( S \) is oriented by the outward-pointing normal.

(a) Give a parametrization of \( S_{\text{sides}} \).

**Solution:**

\[
\Phi(\theta, y) = (3 \cos \theta, y, 3 \sin \theta); \quad 0 \leq \theta \leq 2\pi, \quad 1 \leq y \leq 3.
\]

(b) Compute the flux of \( \vec{F} \) through \( S \): \( \iint_S \vec{F} \cdot d\vec{S} \).

(This requires the computation of three double integrals.)
Solution: We first compute \[ \iiint_{S_{\text{side}}} \vec{F} \cdot d\vec{S}. \] We have \( \vec{T}_\theta = -3 \sin \theta \hat{i} + 3 \cos \theta \hat{k} \) and \( \vec{T}_y = \hat{j} \).

Hence, we compute a normal vector to the surface:

\[ \vec{T}_\theta \times \vec{T}_y = -3 \sin \theta \hat{k} - 3 \cos \theta \hat{i}. \]

However, this vector points inward, so we take instead

\[ \vec{T}_y \times \vec{T}_\theta = 3 \cos \theta \hat{i} + 3 \sin \theta \hat{k}. \]

It follows that

\[ \vec{F} \cdot d\vec{S} = (3 \cos \theta \hat{i} - y \hat{j} + 3 \sin \theta \hat{k}) \cdot (3 \cos \theta \hat{i} + 3 \sin \theta \hat{k}) = 9 \cos^2 \theta + 9 \sin^2 \theta = 9. \]

Therefore, we have

\[ \iiint_{S_{\text{side}}} \vec{F} \cdot d\vec{S} = 9 \iiint_{D} d\theta dy = 9 \cdot \text{area}(D) = 9 \cdot (\pi \cdot 2^2) = 36\pi. \]

Next, we compute \( \iiint_{S_{\text{right cap}}} \vec{F} \cdot d\vec{S}. \) In this case, an outward unit normal is \( \vec{n} = \hat{j}. \) As such, we have \( \vec{F} \cdot \vec{n} = (x \hat{i} - 3 \hat{j} + z \hat{k}) \cdot \hat{j} = -3. \) We obtain

\[ \iiint_{S_{\text{right cap}}} \vec{F} \cdot d\vec{S} = -3 \iiint_{S_{\text{right cap}}} dS = -3 \cdot \text{area}(S_{\text{right cap}}) = -27\pi. \]

Similarly, the outward unit normal to \( S_{\text{left cap}} \) is \( \vec{n} = -\hat{j}, \) which gives

\[ \vec{F} \cdot \vec{n} = (x \hat{i} - \hat{j} + z \hat{k}) \cdot (-\hat{j}) = 1, \] which yields

\[ \iiint_{S_{\text{left cap}}} \vec{F} \cdot d\vec{S} = \iiint_{S_{\text{left cap}}} dS = \text{area}(S_{\text{left cap}}) = 9\pi. \]

In conclusion, we have

\[ \iiint_{S} \vec{F} \cdot d\vec{S} = 36\pi - 27\pi + 9\pi = 18\pi. \]

Supplementary Problem: Attempt this problem only if you have completed problems 1 - 5.

Suppose that \( D \subset \mathbb{R}^2 \) is a simply connected region to which Green’s Theorem applies and that \( P(x, y) \) and \( Q(x, y) \) are \( C^1 \) scalar functions. (Note: simply connected means that “\( D \) can be continuously deformed to a single point”—i.e., it has no “holes”). Prove that

\( \int \limits_{\partial D} P \, dx + Q \, dy = 0 \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \forall (x, y) \in D. \)
Solution: Suppose that $\partial P/\partial y = \partial Q/\partial x$ for all $(x, y) \in D$. Green’s Theorem implies that

$$\int_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = 0.$$  

Now, suppose, for all simple closed curves $C$, that $\int_C P \, dx + Q \, dy = 0$. Suppose, by way of contradiction, that there exists a point $\alpha$ at which $\partial Q/\partial x - \partial P/\partial y > 0$. Then, since partials are continuous, there exists a neighborhood $N$ of $\alpha$ such that $\partial Q/\partial x - \partial P/\partial y > 0$ at all points in $N$. Green’s Theorem implies that

$$\int_{\partial N} P \, dx + Q \, dy = \iint_N \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy > 0,$$

contrary to hypothesis. Similarly, if we assume that $\partial Q/\partial x - \partial P/\partial y < 0$, we reach a contradiction.