1. **(5 points):** Give the cosine of the angle between the vectors

\[ \mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}. \]

**Solution:** We have

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2 + 1 + 2}{\sqrt{1 + 1 + 4} \sqrt{1 + 4 + 1}} = \frac{5}{6}.
\]

2. **(15 points):** Lines. Find parametric equations for the line \( L \) through the point \((1, 2, 3)\) that is parallel to the plane \( Q : x + y + z = 2 \) and perpendicular to the line

\[ L_1 : x = 1 + 2t; \quad y = 1 - 2t; \quad z = t. \]

**Solution:** The vector \( \mathbf{n} = \langle 1, 1, 1 \rangle \) is normal to \( Q \); the vector \( \mathbf{v}_1 = \langle 2, -2, 1 \rangle \) is the direction vector of \( L_1 \). It follows that \( L \) has direction vector

\[
\mathbf{n} \times \mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i}(1 + 1) - \mathbf{j}(1 - 2) + \mathbf{k}(-2 - 2) = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}.
\]

It follows that \( L \) has parametric equations

\[ x = 1 + 3t; \quad y = 2 + t; \quad z = 3 - 4t, \quad t \in \mathbb{R}. \]

3. **(15 points):** Planes. Find the equation of the plane that passes through the point \((1, 2, 3)\) and contains the line

\[ L : x = 3t; \quad y = 1 + t; \quad z = 2 - t, \quad t \in \mathbb{R}. \]

**Solution:** The line has direction vector \( \mathbf{v} = \langle 3, 1, -1 \rangle \). We set \( t = 0 \) in the line to obtain a second point in the plane, \((0, 1, 2)\); therefore, a second vector in the plane is \( \mathbf{u} = (1, 2, 3) - (0, 1, 2) = \langle 1, 1, 1 \rangle \). A normal vector to the plane is

\[
\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i}(1 + 1) - \mathbf{j}(3 + 1) + \mathbf{k}(3 - 1) = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.
\]
Hence, an equation for the plane is
\[ 2(x - 1) - 4(y - 2) + 2(z - 3) = 0 \iff 2x - 2 - 4y + 8 + 2z - 6 = 0 \]
\[ \iff 2x - 4y + 2z = 0 \iff x - 2y + z = 0. \]

4. (15 points): Tangent plane. Find the equation of the plane tangent to \( z = x^2y + 2xy^3 \) at the point \( (1, 1, 3) \).

Solution: With \( z = f(x, y) = x^2y + 2xy^3 \), we compute
\[ f_x = 2xy + 2y^3 \iff f_x(1, 1) = 4; \quad f_y = x^2 + 6xy^2 \iff f_y(1, 1) = 7. \]
It follows that
\[ 4(x - 1) + 7(y - 1) = z - 3 \iff 4x - 4 + 7y - 7 = z - 3 \iff 4x + 8y - z = 8. \]

5. (15 points): Chain rule.

(a) (10 points): Let \( z = e^{xy^2}, \quad x = t \cos t, \quad y = t \sin t \). Compute \( dz/dt \). You may leave your answer in the variables \( x, y, \) and \( t \).

Solution: The chain rule gives
\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (y^2e^{xy^2})(-t \sin t + \cos t) + (2xye^{xy^2})(t \cos t + \sin t). \]

(b) (5 points): Suppose that
\[ z = F(x, y), \quad x = G(u, v), \quad u = H(t), \quad v = I(t). \]
Write down the form of the chain rule that you would use to compute \( dz/dt \). (Use a tree diagram to show variable dependencies.)

Solution: We have
\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}. \]

6. (15 points): Gradient and directional derivative. Let \( F(x, y, z) = 2x^3y - 3y^2z \).

(a) (10 points): Compute the directional derivative of \( F \) at \( (1, -1, 1) \) in the direction of the vector \( \vec{u} = \langle 2, 3, 6 \rangle \). Is \( F \) increasing or decreasing at this point?

Solution: We have \( \nabla F = 6x^2y \vec{i} + (2x^3 - 6yz) \vec{j} - 3y^2 \vec{k} \); it follows that \( \nabla F(1, -1, 1) = -6\vec{i} + 8\vec{j} - 3\vec{k} \). Furthermore, we have \( \frac{\vec{u}}{||\vec{u}||} = \frac{\langle 2, 3, 6 \rangle}{\sqrt{4 + 9 + 36}} = \frac{\langle 2, 3, 6 \rangle}{7} \). We compute
\[ D_{\vec{u}}F(1, -1, 1) = \langle -6, 8, -3 \rangle \cdot \frac{\langle 2, 3, 6 \rangle}{7} = \frac{-12 + 24 - 18}{7} = -\frac{6}{7}. \]
Since \( D_{\vec{u}}(1, -1, 1) < 0 \), the function is decreasing.
(b) **(5 points)**: In what direction from \((1, -1, 1)\) is the directional derivative of \(F\) a maximum? Your answer need not be a unit vector.

**Solution**: The direction of maximum increase is \(\nabla F(1, -1, 1) = (-6, 8, -3)\).

7. **(15 points)**: **Local extrema.** Let \(f(x, y) = x^3 + y^3 - 3x - 12y + 20\). Find the point(s) \((x, y)\) at which \(f(x, y)\) has a local maximum, minimum, or saddle.

**Solution**: We have

\[
\begin{align*}
    f_x &= 3x^2 - 3 = 0 \implies x = \pm 1; \quad f_y = 3y^2 - 12 = 0 \implies y = \pm 2.
\end{align*}
\]

The critical points are \(\{(1, 2), (1, -2), (-1, 2), (-1, -2)\}\). In order to apply the second derivative test, we require

\[
\begin{vmatrix}
    6x & 0 \\
    0 & 6y
\end{vmatrix} = 36xy.
\]

We test the points as follows:

\((1, 2)\): \(f_{xx} > 0\) and \(D > 0\) \(\implies (1, 2)\) is a local minimum.

\((1, -2)\): \(D < 0\) \(\implies (1, -2)\) is a saddle point.

\((-1, 2)\): \(D < 0\) \(\implies (-1, 2)\) is a saddle point.

\((-1, -2)\): \(f_{xx} < 0\) and \(D > 0\) \(\implies (-1, -2)\) is a local maximum.

8. **(15 points)**: **Absolute extrema.** Find the absolute maximum of \(f(x, y) = 2x + 3y + 1\) on the triangular region \(D\) in the first quadrant bounded by:

\[
L_1 : x = 0 \, (y\text{-axis}); \quad L_2 : y = 0 \, (x\text{-axis}); \quad L_3 : x + y = 2.
\]

Clearly explain your answer.

**Solution**: Since \(f_x = 2 \neq 0\) and \(f_y = 3 \neq 0\), there are no critical points inside \(D\). Hence, the extrema of \(f(x, y)\) lie on the boundary of \(D\).

\[L_1:\] We have \(u_1(y) = f(0, y) = 3y + 1\). Since \(u_1'(y) = 3 \neq 0\), the extrema on \(L_1\) occur at the endpoints: \(f(0, 0) = 1\) and \(f(0, 2) = 7\).

\[L_2:\] We have \(u_2(x) = f(x, 0) = 2x + 1\). Since \(u_2'(x) = 2 \neq 0\), the extrema on \(L_2\) occur at the endpoints: \(f(0, 0) = 1\) and \(f(2, 0) = 5\).

\[L_3:\] We have \(u_3(x) = f(x, 2 - x) = 2x + 3(2 - x) + 1 = 7 - x\). Since \(u_3'(x) = -1\), the extrema on \(L_3\) occur at the endpoints: \(f(0, 2) = 7\) and \(f(2, 0) = 5\).

It follows that the absolute maximum of \(f(x, y)\) on \(D\) is \(f(0, 2) = 7\).
9. **(15 points): Lagrange multipliers.** Use Lagrange multipliers to find the point(s) \((x, y, z)\) with \(x, y, z \geq 0\) which maximize \(f(x, y, z) = xyz^2\) subject to the constraint \(x + 2y + 2z = 6\).

**Solution:** Let \(g(x, y, z) = x + 2y + 2z\). We have
\[
\nabla f = yz^2 \mathbf{i} + xz^2 \mathbf{j} + 2xyz \mathbf{k} = \lambda (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \lambda \nabla g.
\]
Equating components gives
\[
yz^2 = \lambda, \quad xz^2 = 2\lambda \implies \frac{xz^2}{2} = \lambda, \quad 2xyz = 2\lambda \implies xyz = \lambda.
\]
Now, \(yz^2 = \frac{xz^2}{2}\) implies that \(y = \frac{x}{2}\). Similarly, \(yz^2 = xyz\) implies that \(x = z\). We substitute in the constraint equation to obtain
\[
x + 2\left(\frac{x}{2}\right) + 2x = 4x = 6 \implies x = \frac{3}{2}, \quad y = \frac{3}{4}, \quad z = \frac{3}{2}.
\]

10. **(15 points): Polar coordinates.** Set up an integral in polar coordinates for the volume of the solid \(E\) that lies below the plane \(z = 6\) and above the paraboloid \(z = 3x^2 + 3y^2\) in the first octant. (How do the plane and paraboloid intersect?)

**Solution:** We have
\[
V = \int_0^{\pi/2} \int_0^{\sqrt{2}} (6 - 3r^2 - 3r^2) \, r \, dr \, d\theta = 3 \int_0^{\pi/2} \int_0^{\sqrt{2}} (2 - r^2) \, dr \, d\theta.
\]

11. **(15 points): Rectangular coordinates.** Suppose that \(E\) is the tetrahedron (a polyhedron with four vertices and four triangular faces, three of which meet at each vertex) bounded by the planes
\[
x = y, \quad x + y = 4, \quad y = z, \quad z = 0 \quad (xy\text{-plane}).
\]
Express the integral \(\iiint_E dV\) as an iterated integral with \(dV = dy \, dx \, dz\). Do not evaluate.

(Draw a quick sketch. What is the shadow (projection) of \(E\) on the \(xz\)-plane?)

**Solution:** We have
\[
\iiint_E dV = \int_0^2 \int_z^{4-z} \int_x^{4-x} dy \, dx \, dz.
\]

12. **(15 points): Cylindrical coordinates.** Convert the integral
\[
\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} y \sqrt{x^2 + y^2} \, dz \, dy \, dx
\]
from rectangular to cylindrical coordinates. Do not evaluate.

**Solution:** We have

\[
\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} y\sqrt{x^2+y^2} \, dz \, dy \, dx = \int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r \sin \theta \cdot r \cdot r \, dz \, dr \, d\theta
\]

\[= \int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r^3 \sin \theta \, dz \, dr \, d\theta.\]

13. (15 points): Spherical coordinates. Use spherical coordinates to evaluate the triple integral

\[
\iiint_{E} xz \, dV
\]

where E is the solid region that lies within the sphere \(x^2 + y^2 + z^2 = 4\) and below the cone \(z = \sqrt{x^2 + y^2}\) in the first octant.

**Solution:** We have

\[
\int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} (\rho \sin \phi \cos \theta)(\rho \cos \phi)(\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{2} \rho^4 \sin^2 \phi \cos \phi \cos \theta \, d\rho \, d\phi \, d\theta
\]

\[= \left( \int_{0}^{\frac{\pi}{2}} \cos \theta \, d\theta \right) \left( \int_{0}^{\frac{\pi}{2}} \sin^2 \phi \cos \phi \, d\phi \right) \left( \int_{0}^{2} \rho^4 \, d\rho \right) = \left( \sin \theta \right) \left( \frac{\pi}{2} \right) \left( \frac{32}{5} \right) \left( 1 - \left(\frac{\sqrt{2}}{2}\right)^2 \right) = \frac{32}{15} \left( 1 - \frac{\sqrt{2}}{4} \right) = \frac{32}{15} - \frac{8\sqrt{2}}{15} = \frac{8}{15}(4 - \sqrt{2}).\]

14. (15 points): Change of variable in a double integral. Compute the double integral

\[
\iint_{R} \frac{x-y}{x+y} \, dA
\]

over the square \(R\) with vertices \((0,2), (1,1), (2,2), (1,3)\). Use a suitable change of variable.

**Solution:** Let \(u = x - y\) and \(v = x + y\). We have

\[
\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.
\]

It follows that

\[
\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2} = \frac{1}{2}.
\]
Therefore, we have

\[
\iint_R \frac{x - y}{x + y} \, dA = \frac{1}{2} \int_S u \, dv.
\]

It remains to determine \( S \) and to do the evaluation. Since the transformation is linear, it will map the square to a quadrilateral whose vertices are:

\[
(0, 2) \mapsto (-2, 2); \quad (1, 1) \mapsto (0, 2); \quad (2, 2) \mapsto (0, 4); \quad (1, 3) \mapsto (-2, 4).
\]

The quadrilateral \( S \) is in fact, a square. We obtain

\[
\iint_R \frac{x - y}{x + y} \, dA = \frac{1}{2} \int_2^4 \int_{-2}^0 \frac{u^2}{v} \, du \, dv = \frac{1}{2} \int_2^4 \left( u^2 \bigg|_{-2}^0 \right) \, dv = - \int_2^4 \frac{dv}{v} = -(\ln 4 - \ln 2) = -(2 \ln 2 - \ln 2) = -\ln 2.
\]