1. (20 points): Volume.

(a) (10 points): Set up a double integral for the volume of the solid bounded above by the paraboloid \( z = x^2 + 3y^2 \), below by the plane \( z = 0 \), and laterally (on the sides) by the parabolic cylinders \( y^2 = x \) and \( y = x^2 \). Do not evaluate.

Solution:

\[
\int_{0}^{1} \int_{x^2}^{\sqrt{x}} (x^2 + 3y^2) \, dy \, dx = \int_{0}^{1} \int_{\sqrt{x}}^{\sqrt[3]{y/3}} (x^2 + 3y^2) \, dx \, dy.
\]

(b) (10 points): Set up a triple integral for the volume of the solid in the first octant bounded by the cylinder \( y^2 + z^2 = 9 \) and the planes \( x = 0 \) and \( x = 3y \). Do not evaluate. (You should view \( D \), the projection of the solid, in the \( yz \)-plane.)

Solution: I was looking for one of

\[
\int_{0}^{3} \int_{0}^{\sqrt{9-z^2}} \int_{0}^{3y} dx \, dz \, dy = \int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{0}^{3y} dx \, dz \, dy.
\]

Alternatives include, for example:

\[
\int_{0}^{9} \int_{x/3}^{3} \int_{0}^{\sqrt{9-y^2}} dz \, dy \, dx = \int_{0}^{3} \int_{0}^{3y} \int_{0}^{\sqrt{9-y^2}} dx \, dz \, dy.
\]

2. (18 points): Let \( R \) be the triangular region in the \( xy \)-plane bounded by \( x = 0 \), \( y = 2x \), and \( y = 4 \). Evaluate

\[
\int \int_{R} e^{x/y} \, dA.
\]

Solution:

\[
\int_{0}^{4} \int_{0}^{y/2} e^{x/y} \, dx \, dy = \int_{0}^{4} \left( ye^{x/y} \right)_{0}^{y/2} \, dy = \int_{0}^{4} (ye^{1/2} - y) \, dy = (e^{1/2} - 1) \int_{0}^{4} y \, dy
\]

\[
= (e^{1/2} - 1) \left( \frac{y^2}{2} \right)_{0}^{4} = 8(e^{1/2} - 1).
\]
3. (15 points): Suppose that $E$ is a solid bounded by

$$x = 2, \quad y = 0, \quad z = 1, \quad x - y + 2z = 2.$$ 

Express the integral

$$\iiint_E f(x, y, z) \, dV$$

as an iterated integral with $dV = dy \, dx \, dz$. (You should view $D$, the projection of $E$, in the $xz$-plane.)

**Note:** This problem was not graded because it was originally stated with $x + y - 2z = 2$ in place of $x - y + 2z = 2$. The constraints $x + y - 2z = 2, x = 2, y = 0, z = 1$ do not define a bounded region. The constraints $x - y + 2z = 2, x = 2, y = 0, z = 1$ define a tetrahedron in the first quadrant, one of whose faces is a triangle in the $xz$-plane with vertices $(2, 0, 0)$, $(2, 0, 1)$, and $(0, 0, 1)$. The remaining vertex of the tetrahedron is the point $(2, 2, 1)$.

**Solution:**

$$\int_0^1 \int_{2-2z}^{x+2z-2} \int_0^y f(x, y, z) \, dy \, dx \, dz.$$

4. (15 points): Let $R$ be the region in the $xy$-plane that lies to the left of the $y$-axis and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Set up the integral

$$\iint_R (x + y) \, dA$$

using polar coordinates.

**Solution:**

$$\int_{\pi/2}^{3\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r \, dr \, d\theta = \int_{\pi/2}^{3\pi/2} \int_1^2 r^2 (\cos \theta + \sin \theta) \, dr \, d\theta.$$

5. (15 points): Let $E$ be the solid in the first octant that lies beneath the paraboloid $z = 8 - x^2 - y^2$ and above the paraboloid $z = x^2 + y^2$. Set up the integral

$$\iiint_E x^2 \sqrt{x^2 + y^2} \, dV$$

using cylindrical coordinates. Do not evaluate. (To determine $D$, the projection of $E$ on the $xy$-plane, find the intersection of the paraboloids.)

**Solution:**

$$\int_0^{\pi/2} \int_0^2 \int_0^{8-r^2} r^2 \cos^2 \theta \cdot r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 \int_0^{8-r^2} r^4 \cos^2 \theta \, dz \, dr \, d\theta.$$
6. **(17 points): Evaluate** the integral by changing to spherical coordinates.

\[
\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_0^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)z \, dz \, dx \, dy.
\]

**Solution:**

\[
\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_0^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)z \, dz \, dx \, dy = \int_0^\pi \int_0^{\pi/2} \int_0^4 \rho^2 \cdot \rho \cos \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \left( \int_0^\pi d\theta \right) \left( \int_0^{\pi/2} \sin \phi \cos \phi \, d\phi \right) \left( \int_0^4 \rho^5 \, d\rho \right) = \pi \cdot \left( \int_{u=0}^1 u \, du \right) \cdot \left( \frac{\rho^6}{6} \right)_0^4
\]

\[
= \pi \cdot \left( \frac{u^2}{2} \right)_0^1 \cdot \frac{4^6}{6} = \frac{4^6 \pi}{12} = \frac{4^5 \pi}{3}.
\]