1. §11.3, #21 (5 points): Determine whether the series is convergent or divergent. Be sure to justify your work.

\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n}. \]

You may assume that \( f(x) = \frac{1}{x \ln x} \) is continuous, positive, and decreasing on \([2, \infty)\).

**Solution:** We use the Integral Test (IT). We compute

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t \to \infty} \ln |u| \bigg|_{\ln 2}^{\ln t} \\
= \lim_{t \to \infty} (\ln |\ln t| - \ln |\ln 2|) = +\infty.
\]

It follows that the improper integral diverges. Therefore, the IT implies that the series diverges.
2. §11.4, #11 (5 points): Determine whether the series is convergent or divergent. Justify your conclusion by using the comparison test. (CT)

\[ \sum_{n=1}^{\infty} \frac{n-1}{n^{4n}}. \]

**Solution:** For large \( n \), the terms are like \( \frac{n}{n^{4n}} = \frac{1}{4n} \). I.e., the terms are like terms in a geometric series with \( r = 1/4 < 1 \). Therefore, we expect the series to converge. To show this using the **Comparison Test** (CT), we note that

\[ a_n = \frac{n-1}{n^{4n}} < \frac{n}{n^{4n}} = \frac{1}{4n} = b_n. \]

Therefore we have

\[ \sum \frac{n-1}{n^{4n}} < \sum \frac{1}{4n}. \]

Since the latter sum converges, so does the former.

3. §11.4, #25 (5 points): Determine whether the series is convergent or divergent. Justify your work by using the limit comparison test. (LCT)

\[ \sum_{n=1}^{\infty} \frac{1 + n + n^2}{\sqrt{1 + n^2 + n^6}}. \]

(What do the terms \( a_n \) look like when \( n \) is big? Use your analysis to find a simple \( b_n \) for the comparison.)

**Solution:** For large \( n \), the terms are like \( \frac{n^2}{\sqrt{n^6}} = \frac{n^2}{n^{3/2}} = \frac{1}{n} \). Therefore, we apply the **limit comparison test** (LCT) with \( a_n = \frac{1 + n + n^2}{\sqrt{1 + n^2 + n^6}} \) and \( b_n = 1/n \). We have have

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1+n+n^2}{\sqrt{1+n^2+n^6}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n + n^2 + n^3}{\sqrt{1 + n^2 + n^6}} \cdot \frac{1}{\frac{1}{n^{9/2}}} = \lim_{n \to \infty} \frac{n + 1 + \frac{1}{n}}{\sqrt{\frac{1}{n^5} + \frac{1}{n^4} + 1}} = 1 > 0.
\]

Since the limit is finite and positive, we may apply the LCT:

\[ \sum \frac{1}{n} \text{ diverges} \implies \sum \frac{1 + n + n^2}{\sqrt{1 + n^2 + n^6}} \text{ diverges.} \]