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The Critical Exponent of Doubly Singular Parabolic Equations[1]

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In this paper we study the Cauchy problem of doubly singular parabolic equations $u_t = \text{div}(|\nabla u|^\sigma \nabla u^m) + t^s|x|^\theta u^p$ with non-negative initial data. Here $-1 < \sigma \le 0$, $m > \max\{0, 1-\sigma-(\sigma+2)/N\}$ satisfying $0 < \sigma + m \le 1$, $p > 1$, and $s \ge 0$. We prove that if $\theta > \max\{-(\sigma+2), (1+s)[N(1-\sigma-m)-(\sigma+2)]\}$, then $p_c = (\sigma+m)+(\sigma+m-1)s+[(\sigma+2)(1+s)+\theta]/N > 1$ is the critical exponent; i.e, if $1 < p \le p_c$ then every non-trivial solution blows up in finite time. But for $p > p_c$ a positive global solution exists. © 2001 Academic Press

Key Words: doubly singular parabolic equation; critical exponent; blow up.

1. INTRODUCTION

In this paper we study critical exponent of quasilinear parabolic equations

$$
\begin{aligned}
&u_t = \text{div}\big(|\nabla u|^\sigma \nabla u^m\big) + t^s|x|^\theta u^p, && x \in R^N, \quad t > 0,\\
&u(x,0) = u_0(x) \ge, \not\equiv 0, && x \in R^N,
\end{aligned} \tag{1.1}
$$

where $-1 < \sigma \le 0$, $m > \max\{0, 1-\sigma-(\sigma+2)/N\}$ satisfying $0 < \sigma + m \le 1$, $p > 1$, and $s \ge 0$. $u_0(x)$ is a continuous function in R^N. The existence, uniqueness, and comparison principle for the solution to (1.1) had been proved in [11] (for the definition of solution see [11]). Since $0 < \sigma + m \le 1$, (1.1) is a doubly singular problem and does not have finite speed of propagation. Therefore, $u(x,t) > 0$ for all $x \in R^N$ and $t > 0$.

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Because the main interests of this paper are to study the large-time behavior of solution, we assume that the solution u of (1.1) has very mild regularity. In this context, "$u(x,t)$ blows up in finite time" means that $w(t) = \int_\Omega u(x,t)\,dx \to +\infty$ as $t \to T^-$ for some finite time $T > 0$, where Ω is a bounded domain in R^N.

Our main result reads as follows:

THEOREM 1. *Assume that* $s \geq 0$, $p > 1$, $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma+2)/N\}$ *satisfying* $0 < \sigma + m \leq 1$. *If* $\theta > \max\{-(\sigma+2), (1+s)[N(1-\sigma-m) - (\sigma+2)]\}$, *then* $p_c = (\sigma+m) + (\sigma+m-1)s + [(\sigma+2)(1+s)+\theta]/N > 1$ *is the critical exponent*; *i.e, if* $1 < p \leq p_c$ *then every non-trivial solution of* (1.1) *blows up in finite time*, *whereas if* $p > p_c$ *then* (1.1) *has a small non-trivial global solution*.

The study of blow-up for nonlinear parabolic equations probably originates from Fujita [8], where he studied the Cauchy problem of the semilinear heat equation,

$$\begin{aligned} &u_t = \Delta u + u^p, && x \in R^N, \quad t > 0, \\ &u(x,0) = u_0(x) \geq 0, && x \in R^N, \end{aligned} \tag{1.2}$$

where $p > 1$, and obtained the following results:

(a) If $1 < p < 1 + 2/N$, then every nontrivial solution $u(x,t)$ blows up in finite time.

(b) If $p > 1 + 2/N$ and $u_0(x) \leq \delta e^{-|x|^2} (0 < \delta \ll 1)$, then (1.2) admits a global solution.

In the critical case $p = 1 + 2/N$, it was shown by Hayakawa [10] for dimensions N=1, 2 and by Kobayashi *et al*. [12] for all $N \geq 1$ that (1.2) possesses no global solution $u(x,t)$ satisfying $\|u(\cdot,t)\|_\infty < \infty$ for $t \geq 0$. Weissler [24] proved that if $p = 1 + 2/N$, then (1.2) possesses no global solution $u(x,t)$ satisfying $\|u(\cdot,t)\|_q < \infty$ for $t > 0$ and some $q \in [1, +\infty)$. The value $p_c = 1 + 2/N$ is called the critical exponent of (1.2). It plays an important role in studying the behavior of the solution to (1.2).

In the past couple of years there have been a number of extensions of Fujita's results in several directions. These include similar results for other geometries (cones and exterior domains) [4, 5, 13, 15, 16], quasilinear parabolic equations, and systems [1, 2, 5, 7, 9, 14, 18–20, 22, 23]. In particular, the authors of [2] considered degenerate equations on domains with non–compact boundary. There are also results for nonlinear wave equations and nonlinear Schrödinger equations. We refer the reader to the survey papers by Deng and Levine [5] and Levine [13] for a detailed account of this aspect.

When $m = 1$, (1.1) becomes p-Laplacian equations, and the critical exponents were given by the authors of [19, 21, 22]. When $\sigma = 0$, (1.1) becomes the porous media equations, and the critical exponents were studied by the authors of [13, 17, 18, 22].

This paper is organized as follows. In Section 2 we discuss the qualitative behaviors and give some estimates of solutions to the homogeneous problem

$$\begin{aligned} &u_t = \operatorname{div}\left(|\nabla u|^\sigma \nabla u^m\right), && x \in R^N, \quad t > 0, \\ &u(x,0) = u_0(x) \geq, \not\equiv 0, && x \in R^N. \end{aligned} \tag{1.3}$$

In Section 3, for convenience, we first discuss the special case of (1.1): $s = 0$, i.e,

$$\begin{aligned} &u_t = \operatorname{div}\left(|\nabla u|^\sigma \nabla u^m\right) + |x|^\theta u^p, && x \in R^N, \quad t > 0, \\ &u(x,0) = u_0(x) \geq, \not\equiv 0, && x \in R^N, \end{aligned} \tag{1.4}$$

and prove that if $1 < p \leq \tilde{p}_c \stackrel{\Delta}{=} \sigma + m + (\sigma + 2 + \theta)/N$ then every non-trivial solution of (1.4) blows up in finite time. In Section 4 we prove Theorem 1.

Remark. We end this section with a simple but very useful reduction. When we consider the blow-up case, by the comparison principle we need only consider that $u_0(x)$ is radially symmetric and non-increasing, i.e, $u_0(x) = u_0(r)$ with $r = |x|$, and $u_0(r)$ is non-increasing in r. Therefore, the solution of (1.1) is also radially symmetric and non-increasing in $r = |x|$.

2. ESTIMATES OF SOLUTIONS TO (1.3)

In this section we discuss (1.3) for the radially symmetric case; the main results are three propositions.

PROPOSITION 1. *Assume that* $-1 < \sigma \leq 0$ *and* $m > 1 - \sigma - (\sigma + 2)/N$ *satisfy* $0 < \sigma + m \leq 1$.

(i) *If* $\sigma + m < 1$, *then, for any* $c > 0$, *the equation* (1.3) *has a global self-similar solution*,

$$u(x,t) = ct^{-\alpha}(1 + hr^\nu)^{-q},$$

where $\alpha = N/[N(\sigma + m - 1) + \sigma + 2]$, $\beta = 1/[N(\sigma + m - 1) + \sigma + 2]$, $\nu = (\sigma + 2)/(\sigma + 1)$, $q = (\sigma + 1)/(1 - \sigma - m)$, $r = |x|t^{-\beta}$, *and* $h = h(c) = \frac{1}{q\nu}\beta^{1/(1+\alpha)}c^{(1-\sigma-m)/(\sigma+1)}m^{-1/(\sigma+1)}$.

(ii) *If* $\sigma + m = 1$, *then, for any* $c > 0$, *the equation* (1.3) *has a global self-similar solution*,

$$u(x, t) = ct^{-\alpha} \exp\{-hr^{\nu}\},$$

where $\alpha = N/(\sigma + 2)$, $\beta = 1/(\sigma + 2)$, $\nu = (\sigma + 2)/(\sigma + 1)$, $r = |x|t^{-\beta}$, *and* h *satisfies* $(h\nu)^{\sigma+1} = \beta/m$.

This proposition can be verified directly.

PROPOSITION 2. *Assume that* $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma + 2)/N\}$, *such that* $0 < \sigma + m \leq 1$ *and* $u_0(x)$ *is a non-trivial and non-negative continuous function. If* $u_0(x)$ *is a radially symmetric and non-increasing function, then the solution* $u(x, t)$ *of* (1.3) *satisfies*

$$u_t \geq -\frac{\alpha}{t} u \qquad \text{for all } x \in R^N, \qquad t > 0, \tag{2.1}$$

where $\alpha = N/[N(\sigma + m - 1) + \sigma + 2]$.

Proof. Denote $k = (\sigma + m)/(\sigma + 1)$, let $f = (mk^{-(\sigma+1)})^{1/(k\sigma+k-1)}u$ when $\sigma + m < 1$, and let $f = u$ when $\sigma + m = 1$. Then (1.3) can be rewritten as

$$\begin{aligned} &f_t = d\,\mathrm{div}\big(|\nabla f^k|^{\sigma}\, \nabla f^k\big), && x \in R^N, \qquad t > 0,\\ &f(x, 0) = f_0(x) \geq, \not\equiv 0, && x \in R^N, \end{aligned}$$

where $d = 1$ when $\sigma + m < 1$ and $d = mk^{-(\sigma+1)}$ when $\sigma + m = 1$. Let $g = f^k$; then g satisfies the following equation:

$$\begin{aligned} &g_t^{1/k} = d\,\mathrm{div}\big(|\nabla g|^{\sigma}\, \nabla g\big), && x \in R^N, \qquad t > 0,\\ &g(x, 0) = {f_0}^k(x) \geq, \not\equiv 0, && x \in R^N. \end{aligned}$$

Denote $\mu = (1 + \sigma - 1/k)/(\sigma + 1)$ if $\sigma + m < 1$, and let

$$v = \begin{cases} \frac{1}{\mu} g^{\mu} & \text{if } 0 < \sigma + m < 1,\\ \ln g & \text{if } \sigma + m = 1. \end{cases}$$

Case 1. $0 < \sigma + m < 1$. In this case, $d = 1$ and g satisfies

$$\begin{aligned} g_t &= kg\,\mathrm{div}\big(|\nabla v|^{\sigma}\, \nabla v\big) + g^{-1/k}|\nabla g|^{\sigma+2} \geq kg\,\mathrm{div}\big(|\nabla v|^{\sigma}\, \nabla v\big),\\ v_t &= g^{-1/k(\sigma+1)} g_t = kg^{-1/k(\sigma+1)} g^{1-1/k}\,\mathrm{div}\big(|\nabla g|^{\sigma}\, \nabla g\big)\\ &= k\mu v\,\mathrm{div}\big(|\nabla v|^{\sigma}\, \nabla v\big) + |\nabla v|^{\sigma+2}. \end{aligned} \tag{2.2}$$

Denote $w = \operatorname{div}(|\nabla v|^\sigma \nabla v)$, $\partial/\partial r = {}'$ and let $z = -v$; then $z > 0$, $z' > 0$, and

$$\begin{aligned} z_t &= -k\mu z \operatorname{div}\big(|\nabla z|^\sigma \nabla z\big) - |\nabla z|^{\sigma+2} \\ &= -k\mu z\left[(\sigma+1)(z')^\sigma z'' + \frac{N-1}{r}(z')^{\sigma+1}\right] - (z')^{\sigma+2} = k\mu z w - (z')^{\sigma+2}, \\ w &= -\left[(\sigma+1)(z')^\sigma z'' + \frac{N-1}{r}(z')^{\sigma+1}\right], \\ -w_t &= (\sigma+1)(z')^\sigma z_t'' + (\sigma+1)\sigma(z')^{\sigma-1} z_t' z'' + \frac{(N-1)(\sigma+1)}{r}(z')^\sigma z_t', \quad (2.3) \\ z_t' &= k\mu(z'w + zw') - (\sigma+2)(z')^{\sigma+1} z'', \\ z_t'' &= k\mu(wz'' + 2w'z' + w''z) - (\sigma+2)\big[(z')^{\sigma+1} z''' + (\sigma+1)(z')^\sigma (z'')^2\big]. \end{aligned}$$

By a series of calculation we have

$$\begin{aligned} -w_t = k\mu(\sigma+1)\Big[& z(z')^\sigma \triangle w + 2(z')^{\sigma+1} w' + (\sigma+1)(z')^\sigma w z'' \\ & + \sigma(z')^{\sigma-1} z z'' w' + \frac{N-1}{r}(z')^{\sigma+1} w + \frac{N-1}{r}(z')^\sigma z w'\Big] \\ & - (\sigma+1)(\sigma+2)\Big[(z')^{2\sigma+1} z''' + (1+2\sigma)(z')^{2\sigma}(z'')^2 \\ & + \frac{N-1}{r}(z')^{2\sigma+1} z''\Big]. \end{aligned} \quad (2.4)$$

It follows from (2.3) that

$$\begin{aligned} -w' = {} & \sigma(\sigma+1)(z')^{\sigma-1}(z'')^2 + (\sigma+1)(z')^\sigma z''' \\ & - \frac{N-1}{r^2}(z')^{\sigma+1} + \frac{(N-1)(\sigma+1)}{r}(z')^\sigma z''. \end{aligned}$$

Denote $\varepsilon = k\mu(\sigma+1) = k(1+\sigma-1/k)$; substituting the above expression into (2.4) we get

$$\begin{aligned} -w_t &= \varepsilon a(r,\ t)\triangle w + b(r,\ t)w' - \varepsilon w^2 - (\sigma+2) \\ &\quad \times \left[\frac{N-1}{r^2}(z')^{2\sigma+2} - (\sigma+1)(z')^{2\sigma}(z'')^2\right] \\ &= \varepsilon a(r,\ t)\triangle w + b(r,\ t)w' - \varepsilon w^2 + (\sigma+2) \\ &\quad \times \left[(\sigma+1)w(z')^\sigma z'' - \frac{N-1}{r^2}(z')^{2\sigma+2} + (\sigma+1)\frac{N-1}{r}(z')^{2\sigma+1} z''\right] \\ &= \varepsilon a(r,\ t)\triangle w + b(r,\ t)w' - \varepsilon w^2 - (\sigma+2) \\ &\quad \times \left[w^2 + \frac{2(N-1)}{r}(z')^{\sigma+1} w + \frac{N(N-1)}{r^2}(z')^{2\sigma+2}\right], \end{aligned}$$

where $a(r, t), b(r, t)$ are functions produced by $z(r, t)$ and $z'(r, t)$. Taking into account the Cauchy inequality

$$-2\frac{N-1}{r}(z')^{\sigma+1}w \le \frac{N-1}{N}w^2 + \frac{N(N-1)}{r^2}(z')^{2\sigma+2},$$

we have

$$\begin{aligned}-w_t \le\ & k(\sigma+1-1/k)a(r, t)\triangle w + b(r, t)w' \\ & + \left[1 - k(\sigma+1) - \frac{\sigma+2}{N}\right]w^2;\end{aligned}$$

i.e.,

$$\begin{aligned}w_t \ge\ & k(-\sigma-1+1/k)a(r, t)\triangle w - b(r, t)w' \\ & + \left[\frac{\sigma+2}{N} - k(1/k-(\sigma+1))\right]w^2.\end{aligned}$$

Noticing $k = (\sigma+m)/(\sigma+1)$, we have

$$\begin{aligned}w_t \ge\ & k[1/k-(\sigma+1)]a(r, t)\triangle w - b(r, t)w' \\ & + \frac{\sigma+2+N(\sigma+m-1)}{N}w^2.\end{aligned}$$

Let $y(r, t) = -\alpha/t$. It is obvious that $y_t = k[1/k-(\sigma+1)]a(r, t)\triangle y - b(r, t)y' + y^2/\alpha$. Since $y(r, 0) = -\infty$, it follows by the comparison principle that $w \ge -\alpha/t$ (see [3, 11]); i.e, $\operatorname{div}(|\nabla v|^\sigma \nabla v) \ge -\alpha/t$. By (2.2) we have $g_t \ge -k\alpha g/t$. Since $g = f^k$, it follows that $f_t = -\alpha f/t$; i.e.

$$u_t \ge -\frac{\alpha}{t}u.$$

Case 2. $\sigma + m = 1$. Since this is easy to prove, we omit the details here. Q.E.D.

Remark. For the porous media equation, the authors of [3] proved (2.1) for first time, to our knowledge.

PROPOSITION 3. *Under the assumptions of Propositions* 1 *and* 2, *there exist positive constants* δ, b *such that*:

(i) *When* $\sigma + m < 1$, *then*

$$u(x, t) \ge \delta(t-\varepsilon)^{-\alpha}(1+br^\nu)^{-q} \qquad \forall\,|x| > 1, \qquad t > \varepsilon > 0, \tag{2.5}$$

where $r = |x|(t-\varepsilon)^{-\beta}$, α, β, ν, *and* q *are as in Proposition* 1, *and* b *is a positive constant*.

(ii) *When* $\sigma + m = 1$, *then*

$$u(x, t) \geq \delta(t - \varepsilon)^{-\alpha} \exp\{-br^{\nu}\} \qquad \forall\, |x| > 1, \qquad t > \varepsilon > 0, \tag{2.6}$$

where $r = |x|(t - \varepsilon)^{-\beta}$, α, β, *and* ν *are as in Proposition* 1, *and* b *is a positive constant.*

Proof. In view of Propositions 1 and 2, and using a method similar to that of [21], one can prove Proposition 3. Here we give only the sketch of the proof for the case $\sigma + m < 1$.

Step 1. By use of the methods of Chap. 6 of [6] we can prove the following comparison lemma:

LEMMA 1. *Let* $0 \leq \tau < +\infty$ *and* $S = \{x \in R^N, |x| > 1\} \times [\tau, +\infty)$. *Assume that* v, w *are non-negative functions satisfying*

$$\begin{aligned} &v_t = \operatorname{div}(|\nabla v|^{\sigma} \nabla v^m), && w_t = \operatorname{div}(|\nabla w|^{\sigma} \nabla w^m) \qquad \text{in } S, \\ &v(x, t) \leq w(x, t), && |x| = 1, \qquad \tau < t < +\infty, \\ &v(x, \tau) \leq w(x, \tau), && |x| \geq 1. \end{aligned}$$

Then

$$v(x, t) \leq w(x, t) \qquad \text{in } S.$$

Step 2. From Proposition 1 we have that problem (1.3) has the similarity solutions

$$U_{\mu}(x, t) = \mu^{\rho} U(\mu x, t), \qquad \rho = (\sigma + 2)/(1 - \sigma - m),$$

where $\mu > 0$ is a parameter, and

$$U(x, t) = U_1(x, t) = t^{-\alpha}(1 + hr^{\nu})^{-q}, \; r = |x| t^{-\beta}.$$

In view of Proposition 2 and the expression of $U_{\mu}(x, t)$ we can prove that for suitably small $\mu > 0$, the following holds:

$$\begin{aligned} &U_{\mu}(1, t - \varepsilon) \leq u(1, t) && \text{for } t > \varepsilon, \\ &U_{\mu}(x, t - \varepsilon) = 0 \leq u(x, t) && \text{for } |x| \geq 1, \qquad t = \varepsilon. \end{aligned}$$

By Lemma 1 we see that (2.5) holds. Q.E.D.

3. THE SPECIAL CASE $s = 0$, $1 < p \le \tilde{p}_c$

In this section we study problem (1.4) and prove a blow-up result.

THEOREM 2. *Let σ, m, p, θ be as in Theorem* 1. *If* $1 < p \le \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$, *then every non-trivial solution of* (1.4) *blows up in finite time*.

Let $\phi(x)$ be a smooth, radially symmetric, and non-increasing function which satisfies $0 \le \phi(x) \le 1$, $\phi(x) \equiv 1$ for $|x| \le 1$, and $\phi(x) \equiv 0$ for $|x| \ge 2$. It follows that for $l > 1$, $\phi_l(x) = \phi(x/l)$ is a smooth, radially symmetric, and non-increasing function which satisfies $0 \le \phi_l(x) \le 1$, $\phi_l(x) \equiv 1$ for $|x| \le l$ and $\phi_l(x) \equiv 0$ for $|x| \ge 2l$. It is easy to see that $|\nabla \phi_l| \le C/l$, $|\triangle \phi_l| \le C/l^2$. Let

$$w_l(t) = \int_\Omega u\phi_l \, dx,$$

where $\Omega = R^N \backslash B_1$, with B_1 being the unit ball with center at the origin. We divide the argument into two cases.

Case 1. $m \le 1$. Let $q = (m+\sigma)/(\sigma+1)$ and $v = u^q$; then the equation (1.4) can be written as

$$(v^{1/q})_t = \frac{m}{q^{\sigma+1}} \operatorname{div}\big(|\nabla v|^\sigma \nabla v\big) + |x|^\theta v^{p/q}.$$

Therefore,

$$\begin{aligned} \frac{dw_l}{dt} &= \frac{m}{q^{\sigma+1}} \int_\Omega \operatorname{div}\big(|\nabla v|^\sigma \nabla v\big)\phi_l \, dx + \int_\Omega |x|^\theta v^{p/q} \phi_l \, dx \\ &\ge -\frac{m}{q^{\sigma+1}} \omega_N \int_1^{2l} |v'|^{\sigma+1} |\phi_l'| r^{N-1} \, dr + \int_\Omega |x|^\theta v^{p/q} \phi_l \, dx. \end{aligned}$$

By direct computation we have

$$\int_1^{2l} |v'|^{\sigma+1} |\phi_l'| r^{N-1} \, dr \le \left(\int_1^{2l} |v'| |\phi_l'| r^{N-1} \, dr \right)^{\sigma+1} \left(\int_1^{2l} r^{N-1} |\phi_l'| \, dr \right)^{-\sigma},$$

$$\int_1^{2l} |v'| |\phi_l'| r^{N-1} \, dr = \frac{1}{\omega_n} \int_\Omega \nabla v \cdot \nabla \phi_l \, dx \le \frac{1}{\omega_N} \int_\Omega v |\triangle \phi_l| \, dx,$$

$$\begin{aligned} \int_\Omega v |\triangle \phi_l| \, dx &\le \left(\int_\Omega |x|^\theta v^{p/q} \phi_l \, dx \right)^{q/p} \\ &\quad \times \left(\int_\Omega \{ |\triangle \phi_l|^p \phi_l^{-q} |x|^{-\theta q} \}^{1/(p-q)} \, dx \right)^{(p-q)/p}, \end{aligned}$$

$$\left(\int_\Omega \{ |\triangle \phi_l|^p \phi_l^{-q} |x|^{-\theta q} \}^{1/(p-q)} \, dx \right)^{(p-q)/p} = C_1 l^{[N(p-q) - \theta q - 2p]/p},$$

$$\left(\int_1^{2l} r^{N-1} \phi_l' \, dr \right)^{-\sigma} = C_2 l^{-(N-1)\sigma}.$$

In view of $m \le 1$, we have $q \le 1$, and hence $p/q > 1$.

Case 2. $m > 1$. In this case one has

$$\begin{aligned}\frac{dw_l}{dt} &= \int_\Omega \operatorname{div}\big(|\nabla u|^\sigma \nabla u^m\big)\phi_l\,dx + \int_\Omega |x|^\theta u^p \phi_l\,dx\\ &= \int_{\partial\Omega} |\nabla u|^\sigma \frac{\partial u^m}{\partial \eta}\phi_l\,ds - \int_\Omega |\nabla u|^\sigma \nabla u^m \cdot \nabla\phi_l\,dx + \int_\Omega |x|^\theta u^p\phi_l\,dx\\ &\geq -\int_\Omega |\nabla u|^\sigma \nabla u^m \cdot \nabla \phi_l\,dx + \int_\Omega |x|^\theta u^p \phi_l\,dx\\ &\geq -m\omega_N \int_1^{2l} |u'|^{\sigma+1} u^{m-1} |\phi_l'| r^{N-1}\,dr + \int_\Omega |x|^\theta u^p \phi_l\,dx.\end{aligned}$$

By direct computation and using Hölder's inequality one has

$$\begin{aligned}\int_1^{2l} |u'|^{\sigma+1} u^{m-1}|\phi_l'| r^{N-1}\,dr &\leq \left(\int_1^{2l} |u'| r^{N-1}|\phi_l'|\,dr\right)^{\sigma+1}\\ &\quad\times\left(\int_1^{2l} |\phi_l'| u^{-(m-1)/\sigma} r^{N-1}\,dr\right)^{-\sigma},\end{aligned}$$

$$\begin{aligned}\int_1^{2l} |\phi_l'| u^{-(m-1)/\sigma} r^{N-1}\,dr &= \int_\Omega |\nabla \phi_l| u^{-(m-1)/\sigma}\,dx\\ &\leq \left(\int_\Omega |x|^\theta u^p \phi_l\,dx\right)^{-(m-1)/p\sigma}\Big(\int_\Omega \big\{|x|^{\theta(m-1)}\\ &\quad\times |\nabla\phi_l|^{p\sigma}\phi_l^{m-1}\big\}^{1/(m-1+p\sigma)}\,dx\Big)^{(m-1+p\sigma)/p\sigma},\end{aligned}$$

$$\int_1^{2l} |u'| r^{N-1} |\phi_l'|\,dr = -\frac{1}{\omega_N}\int_\Omega u|\triangle\phi_l|\,dx \leq \frac{1}{\omega_N}\int_\Omega u|\triangle \phi_l|\,dx,$$

$$\begin{aligned}\int_\Omega u|\triangle\phi_l|\,dx &\leq \left(\int_\Omega |x|^\theta u^p\phi_l\,dx\right)^{1/p}\\ &\quad\times\left(\int_\Omega \big\{|x|^{-\theta}|\triangle \phi_l|^p \phi_l^{-1}\big\}^{1/(p-1)}\,dx\right)^{(p-1)/p},\end{aligned}$$

$$\begin{aligned}&\left(\int_\Omega \big\{|x|^{\theta(m-1)}|\nabla\phi_l|^{p\sigma}\phi_l^{m-1}\big\}^{1/(m-1+p\sigma)}\,dx\right)^{(m-1+p\sigma)/p\sigma}\\ &\qquad = C_1'\, l^{[\theta(m-1)+N(m-1+p\sigma)-p\sigma]/p\sigma},\end{aligned}$$

$$\left(\int_\Omega \big\{|x|^{-\theta}|\triangle\phi_l|^p \phi_l^{-1}\big\}^{1/(p-1)}\,dx\right)^{(p-1)/p} = C_2'\, l^{[N(p-1)-2p-\theta]/p}.$$

In view of $m > 1$, $0 < m + \sigma \leq 1$, it follows that $0 < -(m-1)/\sigma \leq 1$.

For the above two cases we always have

$$\begin{aligned}\frac{dw_l}{dt} &\geq -C_3\left(\int_\Omega |x|^\theta u^p\phi_l\,dx\right)^{(\sigma+m)/p} l^{-\theta(m+\sigma)/p-2-\sigma+N-N(\sigma+m)/p}\\ &\quad + \int_\Omega |x|^\theta u^p \phi_l\,dx;\end{aligned}$$

i.e.,

$$\frac{dw_l}{dt} \geq \left\{ -C_3 l^{-\theta(\sigma+m)/p-2-\sigma+N-N(\sigma+m)/p} + \left(\int_\Omega |x|^\theta u^p \phi_l \, dx \right)^{(p-\sigma-m)/p} \right\}$$
$$\times \left(\int_\Omega |x|^\theta u^p \phi_l \, dx \right)^{(\sigma+m)/p}. \tag{3.1}$$

By Hölder's inequality we have

$$\int_\Omega |x|^\theta u^p \phi_l \, dx \geq \left(\int_\Omega u \phi_l \, dx \right)^p \left(\int_\Omega |x|^{-\theta/(p-1)} \phi_l \, dx \right)^{-(p-1)}.$$

Hence

$$\int_\Omega |x|^\theta u^p \phi_l \, dx \geq \begin{cases} c\, w_l^p \, l^{\theta-N(p-1)} & \text{if } \theta < N(p-1), \\ c\, w_l^p (\ln l)^{-(p-1)} & \text{if } \theta = N(p-1), \\ c\, w_l^p & \text{if } \theta > N(p-1). \end{cases} \tag{3.2}$$

We now prove Theorem 2.

(i) First we consider the case $\theta < N(p-1)$. It follows from (3.1) and (3.2) that

$$\frac{dw_l}{dt} \geq \{ -C_3 l^{-\theta(\sigma+m)/p-2-\sigma+N-N(\sigma+m)/p}$$
$$+ C_4 {w_l}^{p-(\sigma+m)} l^{[\theta-N(p-1)](p-(\sigma+m))/p} \}$$
$$\times \left(\int_\Omega |x|^\theta u^p \phi_l \, dx \right)^{(\sigma+m)/p}. \tag{3.3}$$

(a) $p < \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$. Under this assumption, one has

$$\{\theta - N(p-1)\}\{p-(\sigma+m)\}/p > N - 2 - \sigma - \{N(\sigma+m) + \theta(m+\sigma)\}/p,$$

and consequently

$$l^{\{\theta-N(p-1)\}\{p-(\sigma+m)\}/p} / l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} \to +\infty$$
$$\text{as } l \to +\infty. \tag{3.4}$$

Using the fact that w_l is an increasing function of l, we find from (3.3) and (3.4) that there exist $\delta > 0$, $l \gg 1$ such that

$$\frac{dw_l}{dt} \geq \delta \int_\Omega |x|^\theta u^p \phi_l \, dx \geq \delta {w_l}^p(t) l^{\theta-N(p-1)} \qquad \forall\, t > 0.$$

Thus w_l, and consequently u, blows up in finite time, since $p > 1$.

(b) $p = \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$. In this case, $\{\theta - N(p - 1)\}\{p - (\sigma + m)\}/p = N - 2 - \sigma - \{N(\sigma + m) + \theta(m + \sigma)\}/p < 0$. If we can prove that

$$\int_\Omega u\phi_l \, dx$$

is a unbounded function of t for some l, then it can be shown that, as in the above case, w_l, and hence u, blows up in finite time. Otherwise, $u(\cdot, t) \in L^1(\Omega)$ for all $t > 0$ and there exists an $M > 0$ such that

$$\|u(t)\|_{L^1(\Omega)} \le M \qquad \text{for all } t > 0. \tag{3.5}$$

We will prove (3.5) is impossible. Suppose the contrary; it is clear from (3.1) that, for the large l, if $\int_\Omega |x|^\theta u^p \, dx < +\infty$ then $dw_l/dt \ge \frac{1}{2}\int_\Omega |x|^\theta u^p \phi_l \, dx$, and if $\int_\Omega |x|^\theta u^p \, dx = +\infty$ then $w_l'(t) \ge 1$. Therefore,

$$w_l'(t) \ge k_l(t) \overset{\Delta}{=} \min\left\{1, \frac{1}{2}\int_\Omega |x|^\theta u^p \phi_l \, dx\right\}, \qquad l \gg 1,$$

$$w_l(t) - w_l(0) \ge \int_0^t k_l(\tau) d\tau.$$

Let $w(t) = \int_\Omega u(x, t) \, dx$ and take $l \to +\infty$ in the above inequality. We obtain

$$w(t) - w(0) \ge \int_0^t k(\tau) d\tau, \tag{3.6}$$

where $k(t) = \min\{1, \frac{1}{2}\int_{R^N} |x|^\theta u^p \, dx\}$. When $\sigma + m < 1$, using (2.5) and by direct computation we have

$$\begin{aligned}\int_\Omega |x|^\theta u^p \, dx &\ge \delta^p (t - \varepsilon)^{-1} \int_{|y| \ge (t-\varepsilon)^{-\beta}} |y|^\theta (1 + b|y|^\nu)^{-qp} dy \\ &\ge c(t - \varepsilon)^{-1}, \qquad t \gg 1.\end{aligned}$$

When $\sigma + m = 1$, using (2.6) and by direct computation we have

$$\begin{aligned}\int_\Omega |x|^\theta u^p \, dx &\ge \delta^p (t - \varepsilon)^{-1} \int_{|y| \ge (t-\varepsilon)^{-\beta}} |y|^\theta \exp\{-b|y|^\nu\} dy \\ &\ge c(t - \varepsilon)^{-1}, \qquad t \gg 1.\end{aligned}$$

In view of (3.6) it yields

$$\lim_{t \to +\infty} w(t) = +\infty;$$

i.e.,

$$\lim_{t \to +\infty} \int_\Omega u(x, t) \, dx = +\infty.$$

This shows that (3.5) is impossible. And hence $u(x, t)$ blows up in finite time.

(ii) Next we consider the case $\theta \geq N(p-1)$. Since $m > 1-\sigma-(\sigma+2)/N$, it follows that $N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p < 0$. Combining (3.2) and (3.1) we find that, for the case $\theta = N(p-1)$,

$$\frac{dw_l}{dt} \geq \left(-C_3 l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} + C{w_l}^{p-(\sigma+m)}(\ln l)^{\frac{(\sigma+m-p)(p-1)}{p}}\right) \times \left(\int_\Omega |x|^\theta u^p \phi_l \, dx\right)^{(\sigma+m)/p},$$

and for the case $\theta > N(p-1)$

$$\frac{dw_l}{dt} \geq \left(-C_3 l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} + C{w_l}^{p-(\sigma+m)}\right) \times \left(\int_\Omega |x|^\theta u^p \phi_l \, dx\right)^{(\sigma+m)/p}.$$

Similar to the arguments of (i) one can prove that w_l, and consequently u, blows up in finite time.

Remark 2.3. The reason for using $\Omega = R^N \backslash B_1$ rather than R^N itself is that if $\theta > 0$, then $\int_{B_1} |x|^{-\theta/(p-1)}\, dx$ may not converge.

4. PROOF OF THEOREM 1

(i) If $p \leq p_c = \sigma + m + (\sigma + m - 1) + [(\sigma+2)(1+s)+\theta]/N$, using the methods similar to those of the last section and the papers [19, 21], it can be proved that every non-trivial solution of (1.1) blows up in finite time. We omit the details.

(ii) If $p > p_c = \sigma + m + (\sigma+m-1)s + [(\sigma+2)(1+s)+\theta]/N$, we shall prove that (1.1) has global positive solutions for the small initial data. By the comparison principle, it is enough to prove this conclusion for the problem (since $s \geq 0$)

$$\begin{aligned} &u_t = \operatorname{div}(|\nabla u|^\sigma \nabla u^m) + (1+t)^s |x|^\theta u^p, && x \in R^N, \quad t > 0, \\ &u(x,0) = u_0(x) \geq 0, && x \in R^N, \end{aligned} \tag{4.1}$$

where the constants m, σ, s, θ, p are as in problem (1.1). We shall deal with the global solutions of (4.1) by using the similarity solutions which take the form

$$u(x,t) = (1+t)^{-\alpha} w(r) \qquad \text{with } r = |x|(1+t)^{-\beta},$$

where $\alpha = \{1+s+\frac{\theta}{\sigma+2}\}/\{p-1-\frac{1-\sigma-m}{\sigma+2}\theta\}$, $\beta = \{(1-\sigma-m)(1+s)+p-1\}/\{(p-1-\frac{1-\sigma-m}{\sigma+2}\theta)(\sigma+2)\}$, and w satisfies the following ODE:

$$m(\sigma+1)|w'|^{\sigma}w''w^{m-1}+m(m-1)w^{m-2}|w'|^{\sigma+2}$$
$$+m\frac{N-1}{r}|w'|^{\sigma}w'w^{m-1}+\alpha w+\beta rw'+r^{\theta}w^{p}=0, \qquad r>0,$$
$$w(0)=\eta>0, \quad |w'|^{\sigma}w'(0)=-\lim_{r\to 0^+}\{r^{\theta+1}w^{p+1-m}(r)/[(N-1)m]\}. \tag{4.2}$$

We call $w(r)$ a solution of (4.2) in $(0,\ R(\eta))$ for some $R(\eta)>0$ if $w(r)>0$ in $(0,\ R(\eta))$, $w\in C^2(0,\ R(\eta))$, and w satisfies the initial condition of (4.2). Under our assumptions it follows that $p>1+(1-\sigma-m)\theta/(\sigma+2)$, $\alpha>0$, $\beta>0$. We observe that a function $\bar{u}(x,t)=(1+t)^{-\alpha}v(|x|(1+t)^{-\beta})$ is an upper solution of the equation (4.1) if and only if $v(r)$ satisfies the following inequality:

$$m(\sigma+1)|v'|^{\sigma}v''v^{m-1}+m(m-1)v^{m-2}|v'|^{\sigma+2}$$
$$+m\frac{N-1}{r}|v'|^{\sigma}v'v^{m-1}+\alpha v+\beta rv'+r^{\theta}v^{p}\le 0, \qquad r>0. \tag{4.3}$$

(1) We first discuss the case $\theta\ge 0$. In this case, we try to find an upper solution of (4.1), i.e., the solution of (4.3).

When $\sigma+m<1$, let $v(r)=\varepsilon(1+br^k)^{-q}$, where $k=(\sigma+2)/(\sigma+1)$, $q=(\sigma+1)/(1-\sigma-m)$, and ε and b are positive constants to be determined later. By direct computation we have

$$v'=-\varepsilon qbkr^{k-1}(1+br^k)^{-q-1},$$
$$v''=\varepsilon q(q+1)b^2k^2r^{2k-2}(1+br^k)^{-q-2}-\varepsilon qbk(k-1)r^{k-2}(1+br^k)^{-q-1}.$$

$v(r)$ satisfies (4.3) if and only if

$$\varepsilon(1+br^k)^{-q}\big[\alpha-mN\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1}\big]+\varepsilon qbkr^k(1+br^k)^{-q-1}$$
$$\times\big[m\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1}-\beta\big]+\varepsilon^p r^{\theta}(1+br^k)^{-qp}\le 0. \tag{4.4}$$

Under our assumptions it follows that $\theta+q(1-p)k=\theta+(1-p)(\sigma+2)/(1-\sigma-m)<0$. There exists $a>0$, such that

$$r^{\theta}(1+br^k)^{q(1-p)}\le a \qquad \text{for all } r\ge 0, \qquad \text{since } \theta\ge 0. \tag{4.5}$$

Choose $b=b(\varepsilon)$ such that

$$\beta=m\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1};$$

i.e.,

$$b=(qk)^{-1}\big(\beta m^{-1}\varepsilon^{1-\sigma-m}\big)^{1/(\sigma+1)}.$$

For this choice of b, (4.4) is equivalent to

$$\alpha - N\beta + r^{\theta}\varepsilon^{p-1}(1+br^k)^{q(1-p)} \le 0. \tag{4.6}$$

By (4.5) we see that (4.6) is true if the following inequality holds:

$$\alpha - N\beta + a\varepsilon^{p-1} \le 0. \tag{4.7}$$

In view of $p > p_c = \sigma + m + (\sigma + m - 1)s + [(\sigma+2)(1+s)+\theta]/N$, it follows that $\alpha < N\beta$. Hence, there exists $\varepsilon_0 > 0$ such that (4.7) holds for all $0 < \varepsilon \le \varepsilon_0$. These arguments show that $v(r) = \varepsilon(1+br^k)^{-q}$ satisfies (4.3) for all $0 < \varepsilon \le \varepsilon_0$. Using the comparison principle we get that the solution $u(x,t)$ of (4.1) exists globally provided that $u(x,0) \le v(|x|) = \varepsilon(1+b|x|^k)^{-q}$. And hence, so does the solution of (1.1).

When $\sigma + m = 1$, let $v(r) = \varepsilon\exp\{-br^k\}$, where $k = (\sigma+2)/(\sigma+1)$, and ε and b are positive constants to be determined later. By direct computation we know that $v(r)$ satisfies (4.3) if and only if

$$\begin{aligned}&\varepsilon\big[\alpha - mN(bk)^{\sigma+1}\big]e^{-br^k}\\ &\quad + \varepsilon bk\big[m(bk)^{\sigma+1} - \beta\big]r^k e^{-br^k} + \varepsilon^p r^{\theta} e^{-pbr^k} \le 0.\end{aligned} \tag{4.8}$$

Since $\theta \ge 0$, there exists $a > 0$ such that

$$r^{\theta}\exp\big\{-(p-1)br^k\big\} \le a \qquad \text{for all } r \ge 0.$$

Choose b such that $\beta = m(bk)^{\sigma+1}$. Then (4.8) holds provided that

$$\alpha - N\beta + a\varepsilon^{p-1} \le 0.$$

Similar to the case $\sigma + m < 1$, we have that the solution $u(x,t)$ of (4.1) exists globally provided that $\varepsilon \ll 1$ and $u(x,0) \le v(|x|) = \varepsilon\exp\{-b|x|^k\}$. And hence, so does the solution of (1.1).

(2) Next we consider the case $\theta < 0$. If $m = 1$, this problem was discussed by [19] for $\sigma = 0$, and by [21] for $\sigma < 0$. In the following we always assume that $m \ne 1$. Our main purpose is to prove that (4.2) has ground state for the small $\eta > 0$. By the standard arguments one can prove that for any given $\eta > 0$, there exists a unique solution w of (4.2), which is twice continuously differentiable in where $w'(r) \ne 0$.

Denote $R(\eta) = \max\{R \mid w(r) > 0 \ \forall\ r \in [0,R)\}$. So $0 < R(\eta) \le +\infty$, and $w(R(\eta)) = 0$ when $R(\eta) < \infty$.

We divide the proof into several lemmas.

LEMMA 2. *The solution $w(r)$ of* (4.2) *satisfies $w'(r) < 0$ in* $(0, R(\eta))$. *In addition, if $R(\eta) = +\infty$ then $w(r) \to 0$ as $r \to +\infty$.*

Proof. We first prove that $w'(r) < 0$ for $0 < r < R(\eta)$ when $\theta + 1 \le 0$. Since $|w'|^\sigma w'(0) = -\lim_{r\to 0^+}\{r^{\theta+1}w^{p+1-m}(r)/[(N-1)m]\} < 0$, one has $w'(r) < 0$ for $r \ll 1$. If there exists $r_0 : 0 < r_0 < R(\eta)$ such that $w'(r) < 0$ in $(0, r_0)$ and $w'(r_0) = 0$, then $(|w'|^\sigma w')' w^{m-1}(r_0) \ge 0$. But by the equation (4.2) we see that

$$m(|w'|^\sigma w')' w^{m-1}(r_0) = -\big(\alpha w(r_0) + r_0{}^\theta w^p(r_0)\big) < 0,$$

a contradiction. When $\theta + 1 > 0$, it follows that $w'(0) = 0$. Using the equation (4.2) one has

$$mN(|w'|^\sigma w')'|_{r=0} = -\big(\alpha w^{2-m}(0) + \lim_{r\to 0^+} r^\theta w^{p+1-m}(r)\big) < 0.$$

Hence $|w'|^\sigma w'(r) < 0$, and consequently $w'(r) < 0$, for all $r \ll 1$. Similar to the case of $\theta + 1 \le 0$ it follows that $w'(r) < 0$ for all $0 < r < R(\eta)$. If $R(\eta) = +\infty$, since $w'(r) < 0$ and $w(r) > 0$ in $(0, +\infty)$, one has $\lim_{r\to+\infty} w(r) = L$. If $L > 0$, an integration of (4.2) gives

$$r^{N-1}\big(m|w'|^\sigma w' w^{m-1} + r\beta w\big) = -\int_0^r \big\{\alpha - N\beta + s^\theta w^{p-1}(s)\big\} s^{N-1} w(s)\, ds,$$

$$\lim_{r\to+\infty} \frac{m|w'|^\sigma w' w^{m-1}}{r} = -\frac{\alpha}{N}L - \frac{A}{N},$$

where

$$A = \begin{cases} L^p & \text{if } \theta = 0, \\ 0 & \text{if } \theta < 0, \\ +\infty & \text{if } \theta > 0. \end{cases}$$

It follows that $\lim_{r\to+\infty} w'(r) = -\infty$, a contradiction. Thus $w(r) \to 0$ as $r \to +\infty$. Q.E.D.

LEMMA 3. *Let $w(r)$ be the solution of* (4.2). *Then for any given small $\eta > 0$ there exists $R_0(\eta) > 0$ which satisfies $\lim_{\eta\to 0^+} R_0(\eta) = +\infty$ and such that*

$$w(r) > 0, \qquad m|w'|^\sigma w'(r) w^{m-1} + \beta r w(r) > 0, \qquad r \in (1, R_0(\eta)). \tag{4.9}$$

Proof. Let $z = \eta - w$; then $z'(r) = -w'(r) > 0$, $0 < z(r) < \eta$, and $z(r)$ satisfies

$$\begin{aligned} &m(\sigma+1)(z')^\sigma z''(\eta-z)^{m-1} - m(m-1)(\eta-z)^{m-2}(z')^{\sigma+2} \\ &\quad + m\frac{N-1}{r}(z')^{\sigma+1}(\eta-z)^{m-1} = \alpha(\eta-z) - \beta r z' + r^\theta(\eta-z)^p, \quad r>0, \\ &z(0)=0,\ \ (z')^\sigma z'(0) = \lim_{r\to 0^+}\{r^{\theta+1}(\eta-z)^{p+1-m}(r)/[(N-1)m]\}. \end{aligned} \tag{4.10}$$

Since $p > p_c$, one has $N\beta > \alpha$. An integration of (4.10) gives

$$\begin{aligned} mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} + \beta r^N z \\ = \int_0^r \left[(N\beta - \alpha)s^{N-1}z + \alpha\eta s^{N-1} + s^{N+\theta-1}(\eta - z)^p\right] ds \\ \le \frac{\alpha\eta}{N} r^N + (\beta - \frac{\alpha}{N}) r^N z(r) + \frac{1}{N+\theta}\eta^p r^{N+\theta}. \end{aligned} \tag{4.11}$$

Since $m \ne 1$ and $-1 < \sigma \le 0$, we know that if $\sigma + m = 1$ then $\sigma < 0$ and $1 < m < 2$. Denote $R_0(\eta) = \min\{R \mid z(R) = \eta - \eta^a\}$, where $a = \frac{1}{2}\min\{1 - \frac{\sigma}{m-1}, p+1\}$ if $\sigma + m < 1$ and $m > 1$, $a = (p+1)/2$ if $\sigma + m < 1$ and $m < 1$, and $a = \frac{1}{2}\min\{\frac{p+2m-3}{m-1}, p+1\}$ if $\sigma + m = 1$. Then $R_0(\eta) > 0$ and $z(r) \le \eta - \eta^a < \eta$ for all $0 < r \le R_0(\eta)$.

We first consider the case $\sigma + m < 1$. From (4.11) it follows that for $0 < r \le R_0(\eta)$

$$\begin{aligned} mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} &< \frac{\alpha\eta}{N} r^N + (\beta - \frac{\alpha}{N})\eta r^N + \frac{1}{N+\theta}\eta^p r^{N+\theta} \\ &= \beta\eta r^N + \frac{1}{N+\theta}\eta^p r^{\theta+N}. \end{aligned}$$

Denote $b = a$ when $m > 1$, and $b = 1$ when $m < 1$. Using $\eta^a \le \eta - z \le \eta$ one has that

$$r^{N-1}(z')^{\sigma+1} < \frac{1}{m}\left\{\beta\eta^{1+(1-m)b} r + \eta^{p+(1-m)b}\frac{1}{N+\theta} r^{\theta+1}\right\}.$$

Since $\sigma + 1 \le 1$, it follows that

$$\begin{aligned} z'(r) &< \left\{\frac{\beta}{m}\eta^{1+(1-m)b} r + \frac{1}{m(N+\theta)}\eta^{p+(1-m)b} r^{\theta+1}\right\}^{1/(\sigma+1)} \\ &\le C_1\left\{(\eta^{1+(1-m)b} r)^{1/(\sigma+1)} + (\eta^{p+(1-m)b} r^{1+\theta})^{1/(\sigma+1)}\right\}. \end{aligned}$$

Integrating this inequality from 0 to $R_0(\eta)$ we have

$$\begin{aligned} \eta \le \eta^a + C_2\Big\{&\eta^{(1+(1-m)b)/(\sigma+1)}\big(R_0(\eta)\big)^{(\sigma+2)/(\sigma+1)} \\ &+ \eta^{(p+(1-m)b)/(\sigma+1)}\big(R_0(\eta)\big)^{(\sigma+\theta+2)/(\sigma+1)}\Big\}. \end{aligned}$$

In view of $a > 1$ and $[p + (1-m)b]/(\sigma + 1) > [1 + (1-m)b]/(\sigma + 1) > 1$, it follows that $R_0(\eta) \longrightarrow +\infty$ as $\eta \longrightarrow 0^+$.

Using $w(R_0(\eta)) = \eta^a$ and $w(r) \geq \eta^a$ for all $0 \leq r \leq R_0(\eta)$, an integration of (4.2) gives, for $0 \leq r < R_0(\eta)$,

$$\begin{aligned} & mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) \\ &\quad = \int_0^r (N\beta - \alpha)s^{N-1} w(s)\, ds - \int_0^r s^{N+\theta-1} w^p(s)\, ds \\ &\quad \geq (N\beta - \alpha) w\big(R_0(\eta)\big) \int_0^r s^{N-1}\, ds - \eta^p \int_0^r s^{N+\theta-1}\, ds \\ &\quad = \eta^a r^N \left(\beta - \frac{\alpha}{N} - \frac{1}{N+\theta} \eta^{p-a} r^\theta \right). \end{aligned}$$

Since $\theta < 0$, $N\beta > \alpha$, and $p > a$, it follows that

$$mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) > 0, \qquad \forall\, r \in \big(1, R_0(\eta)\big). \tag{4.12}$$

Second, we consider the case $\sigma + m = 1$. From (4.11) it follows that, for $0 < r \leq R_0(\eta)$,

$$mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} < \frac{\alpha}{N} r^N (\eta - z) + \frac{1}{N+\theta} \eta^p r^{N+\theta}.$$

Using $\sigma + 1 = 2 - m$ and $1 < m < 2$ we have that

$$z'(r) \leq C\big\{(\eta - z) r^{1/(\sigma+1)} + \eta^{(p+(1-m)a)/(\sigma+1)} r^{(1+\theta)/(\sigma+1)}\big\}. \tag{4.13}$$

Denote $\gamma = [p + (1-m)a]/(\sigma+1)$. Integrating (4.13) from 0 to $R_0(\eta)$ we have

$$\begin{aligned} \eta - \eta^a \leq C\bigg\{ & \eta^a \frac{\sigma+1}{\sigma+2} \big(R_0(\eta)\big)^{(\sigma+2)/(\sigma+1)} + \eta^\gamma \frac{\sigma+1}{\sigma+2+\theta} \big(R_0(\eta)\big)^{(\sigma+2+\theta)/(\sigma+1)} \\ & + \frac{\sigma+1}{\sigma+2} \int_0^{R_0(\eta)} r^{(\sigma+2)/(\sigma+1)} z' \bigg\}. \end{aligned} \tag{4.14}$$

Substituting (4.13) into (4.14) and using the inductive method we have that

$$\begin{aligned} \eta - \eta^a \leq \eta^a \sum_{n=1}^{+\infty} \frac{1}{n!} A^n + C(\sigma+1)(R_0(\eta))^{(\sigma+2+\theta)/(\sigma+1)} \eta^\gamma \\ \times \sum_{n=0}^{+\infty} \frac{1}{\big((n+1)(\sigma+2)+\theta\big) n!} A^n, \end{aligned} \tag{4.15}$$

where $A = C\frac{\sigma+1}{\sigma+2}(R_0(\eta))^{(\sigma+2)/(\sigma+1)}$. In view of $a > 1$ and $\gamma = [p + (1-m)a]/(\sigma+1) > 1$, it follows from (4.15) that $R_0(\eta) \longrightarrow +\infty$ as $\eta \longrightarrow 0^+$. Similar to the case $\sigma + m < 1$, we have that (4.12) holds. The proof of Lemma 2 is completed. Q.E.D.

Now we prove that, for the case $\theta<0$, (4.2) has ground state for small η. Choose $\eta_0 : \eta_0^{p-1} < N\beta-\alpha$ such that (4.9) holds for all $0<\eta\le\eta_0$. Since $p>p_c$, which implies $N\beta>\alpha$, using $\theta<0$, $R_0(\eta)>1, w(s)<\eta$ and integrating (4.2) from $R_0(\eta)$ to $r(R_0(\eta)<r<R(\eta))$ we have

$$\begin{aligned} &mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) \\ &= \left(mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r)\right)\big|_{r=R_0(\eta)} \\ &\quad + (N\beta-\alpha)\int_{R_0(\eta)}^r s^{N-1} w(s)\left[N\beta-\alpha-s^\theta w^{p-1}(s)\right] ds \\ &\ge \int_{R_0(\eta)}^r s^{N-1} w(s)\left[N\beta-\alpha-\eta^{p-1}\right] ds \ge 0. \end{aligned} \tag{4.16}$$

In view of $w(r)>0$ and $w'(r)<0$ for $0<r<R(\eta)$, it follows that $R(\eta)=+\infty$ by (4.16). Therefore (4.2) has a ground state.

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REFERENCES

1. D. Andreucci, New results on the Cauchy problem for parabolic systems and equations with strongly nonlinear sources, Manuscript Math. **77** (1992), 127–159.
2. D. Andreucci and A. F. Tedeev, A Fujita type results for a degenerate Neumann problem in domains with noncompact boundary, *J. Math. Anal. Appl.* **231** (1999), 543–567.
3. D. G. Aronson and Ph. Benilan, Regularite des solutions de l'equation des milieux poreux dans $\mathbf{R}^N$, *C. R. Acad. Sci. Paris* **288** (1979), 103–105.
4. C. Bandle and H. A. Levine, On the existence and nonexistence of global solutions of reaction–diffusion equations in sectorial domains, *Trans. Amer. Math. Soc.* **655** (1989), 595–624.
5. K. Deng and H. A. Levine, The role of critical exponents in blow-up theorems: The sequel, *J. Math. Anal. Appl.* **243** (2000), 85–126.
6. E. DiBenedetto, "Degenerate Parabolic Equations," Springer-Verlag, New York, 1993.
7. M. Escobedo and M. A. Herrero, Boundedness and blow up for a semilinear reaction–diffusion system, *J. Differential Equations* **89** (1989), 176–202.
8. H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. 1* **13** (1996), 109–124.
9. V. A. Galaktionov and H. A. Levine, A general approach to critical Fujita exponents in nonlinear parabolic problems, *Nonlinear Anal.* **34**, No. 7 (1998), 1005–1027.
10. K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic equations, *Proc. Japan Acad.* **49** (1973), 503–525.
11. A. S. Kalashnikov, Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations, *Uspekhi Mat. Nauk* **42**, No. 2 (1987), 135–176 (in Russian). English translation: *Russian Math. Surveys* **42**, No. 2 (1987), 169–222.

12. K. Kobayashi, T. Siaro, and H. Tanaka, On the blowing up problem for semilinear heat equations, *J. Math. Soc. Japan* **29** (1977), 407–424.
13. H. A. Levine, The role of critical exponents in blowup theorems, *SIAM Rev.* **32** (1990), 262–288.
14. H. A. Levine, A Fujita type global existence-global nonexistence theorem for a weakly coupled system of reaction–diffusion equations, *Z. Angew. Math. Phys.* **42** (1990), 408–430.
15. H. A. Levine and P. Meier, A blow up result for the critical exponent in cones, *Israel J. Math.* **67** (1989), 1–8.
16. H. A. Levine and P. Meier, The value of the critical exponent for reaction–diffusion equations in cones, *Arch. Rational Mech. Anal.* **109** (1989), 73–80.
17. Y. W. Qi, On the equation $u_t = \triangle u^\alpha + u^\beta$, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 373–390.
18. Y. W. Qi, Critical exponents of degenerate parabolic equations, *Sci. China Ser. A* **38**, No. 10 (1995), 1153–1162.
19. Y. W. Qi, The critical exponents of parabolic equations and blow-up in R^N, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 123–136.
20. Y. W. Qi and H. A. Levine, The critical exponent of degenerate parabolic systems, *Z. Angew. Math. Phys.* **44** (1993), 249–265.
21. Y. W. Qi and M. X. Wang, Critical exponents of quasilinear parabolic equations, preprint.
22. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdynumov, and A. P. Mikhailov, "Blow-up in Quasilinear Parabolic Equations," Nauka, Moscow, 1987 (in Russian); English translation: de Gruyter, Berlin, 1995.
23. P. Souplet, Finite time blow-up for a nonlinear parabolic equation with a gradient term and applications, *Math. Methods Appl. Sci.* **19** (1996), 1317–1333.
24. F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38** (1981), 29–40.