Improper Integrals

Douglas Meade, Ronda Sanders, and Xian Wu
Department of Mathematics

Overview

The topic of improper integrals involves both definite integrals and limits. While Maple usually can evaluate an improper integral directly as if it were a proper integral, the main objective of this lab is to illustrate how to use it to help you to understand individual steps involved in evaluating improper integrals.

Maple Essentials

Important Maple command introduced in this lab:

Command/Example	Description

`assuming;`
 Examples:
 `solve(abs(x)=a,x) assuming a>0;`
 `limit(x*exp(a*x),x=infinity) assuming a<0;`
 `int(sec(x),x=0..a) assuming a>0 and a<Pi/2;`
	It is used to compute the value of an expression under assumptions. Its general form is: (expression) `assuming` (property);, where (expression) is an expression or input to be evaluated and (property) is an assumed property.

Related course material/Preparation

Recall from §7.8 of the Calculus text that an improper integral can be identified as a definite integral over an infinite interval or with a unbounded integrand within the interval of integration. The skills involved in evaluating an improper integral include identifying its improperness, reformulating the integral as an appropriate limit, finding the integral using the Fundamental Theorem of Calculus, and evaluating the limit. You need to understand and work through those steps, especially when the integral involves parameters.

Activities

We will work through some problems to illustrate how to use Maple to carry out individual steps involved in evaluating improper integrals. As you and your TA work through those activities, feel free to experiment with other problems. At the minimum, you should check your work and answers for some homework problems.

Activity/Example 1: Improper integrals with unbounded integrands

1. Create a graph of $f(x) = \ln x$ over $[0, 5]$ and use a limit to verify that $\displaystyle\int_0^5 \ln x \; dx$ is improper.

2. Reformulate the integral as an appropriate limit and find the new integral using the FTOC.

3. Evaluate the limit to see if the improper integral converges or diverges.

Activity/Example 2: Improper integrals over infinite intervals

1. Reformulate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \; dx$ as two appropriate limits and find the new integrals using the FTOC.

2. Evaluate the limits to see if the improper integral converges or diverges.

Activity/Example 3: Improper integrals with parameters.

For what values of p does $\displaystyle\int_1^{\infty} \frac{1}{x^p} \; dx$ converge?

Working with Maple:

- It is advised to start a Maple session with the `restart` command.
  >`restart;`
- **Example 1:** Define and plot $f(x) = \ln x$.
  >`f:=x->ln(x);`
  >`plot(f(x),x=0..5);`
- It looks like the function is unbounded near $x = 0$. Confirm this with a limit.
  >`limit(f(x),x=0,right);`
- Rewrite the integral as a limit using `Limit` and `Int` (**not** `limit` and `int`).
  >`Limit(Int(f(x),x=a..5),a=0,right);`
- Find the new integral using the FTOC (need integration by parts) and evaluate the limit.
  >`I1:=int(f(x),x=a..5);`
  >`limit(I1,a=0,right);`
- **Example 2:** This integral must be reformulated as the sum of two limits. (why?)
  >`I2:=Limit(Int(x/(x^2+1),x=a..0),a=-infinity) +Limit(Int(x/(x^2+1),x=0..b),b=infinity);`
- To find the new integrals and limits, copy and paste the above command and change `Int` and `Limit` to `int` and `limit` when needed.
  >`I2:=Limit(int(x/(x^2+1),x=a..0),a=-infinity) +Limit(int(x/(x^2+1),x=0..b),b=infinity);`
  >`I2:=limit(int(x/(x^2 +1),x=a..0),a=-infinity) +limit(int(x/(x^2+1),x=0..b),b=infinity);`
- To see why this improper integral is undefined and hence diverges, check each limit individually.
  >`I21:=limit(int(x/(x^2+1),x=a..0),a=-infinity);`
  >`I22:=limit(int(x/(x^2+1),x=0..b),b=infinity);`
  >`I2:=I21+I22;`
- You may simply use Maple to check your answers without going through individual steps.
  >`I1:=int(f(x),x=0..5);`
  >`I2:=int(x/(x^2+1),x=-infinity..infinity);`
- **Example 3:** Integrals involving parameters need extra care.
  >`I3:=int(1/x^p,x=1..infinity);`
- Maple is able to find the integral using the FTOC but is unable to come up with a value of the limit. The problem is that the value depends on the parameter p so we have to consider three different cases of $p > 1$, $p < 1$, and $p = 1$. Can you see why?
  >`I3 assuming p>1;`
  >`I3 assuming p<1;`
  >`I3 assuming p=1;`
- When $p = 1$, the general limit originally produced by Maple is of no use because both numerator and denominator are zero when $p = 1$. We need to redo the integral in this case.
  >`I3:=int(1/x,x=1..infinity);`
- This is easy to see, as $\int \frac{1}{x}\, dx = \ln|x|$. In conclusion, the improper integral $\int_{1}^{\infty} \frac{1}{x^p}\, dx$ diverges if $p \le 1$ and converges to $\frac{1}{p-1}$ if $p > 1$. (You need this fact in Chapter 10).

Activity 4:

Repeat Activity/Example 1, 2, and 3 for the following 3 integrals, respectively:

$$1)\int_{\pi/2}^{\pi} \csc(x)\, dx \qquad 2)\int_{-\infty}^{\infty} xe^{-x^2}\, dx \qquad 3)\int_{e}^{\infty} \frac{1}{x(\ln(x))^p}\, dx \qquad$$ (Ex.36, 13, and 58 from the text, pages 515-516, respectively)

Assignment

Finish lab activities and lab instructor will give assignment for each section Please show work and steps.