Sequences and Series

Douglas Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

Overview

Finding limits of sequences and series can be difficult and Maple in general can do it far better than most of us. We will illustrate in this lab how to use Maple to help us with such problems. Ways to generate sequences and series in Maple are also introduced.

Maple Essentials

- New Maple commands introduced in this lab:

Command/Example	Description

`seq(f(n), n=i..j);`
 Example:
 `a:=seq(n!,n=1..8); a[8];`
	It creates a finite sequence of values $f(i)$, $f(i+1)$, $\cdots$, and $f(j)$, where $f(n)$ is a function and $i \leq j$ are integers.

`seq([n,f(n)], n=i..j);`
 Example:
 `pt:=seq([n,1/n],n=1..8); pt;`
	Similarly, it creates a finite sequence of points $[i, f(i)]$, $[i+1, f(i+1)]$, $\cdots$, and $[j, f(j)]$ on the graph of $y = f(x)$, where $i \leq j$ are integers.

`sum(f(n), n=i..j)`
 Examples:
 `sum(n^2, n=-1..10);`
 `sum((1/2)^n,n=0..infinity);`
 `Sum((1/2)^n,n=0..infinity);`
 `f:=x->sum(x^n/n!,n=0..infinity);`
 `PS:=n->sum(x^k/k!,k=0..n);`
	It creates and evaluates a finite or infinite sum, that is, series $\sum_{n=i}^{j} f(n)$, where `f(n)` is a maple function or expression and $i \leq j$ can be integers, variables, or infinity. For a finite or convergent infinite series, it automatically evaluates the sum and returns a value or formula. If you don't want the automatic evaluation, use `Sum` instead of `sum`.

`for n from i to j do...end do;`
 Examples:
 `for n from 3 to 10 do c[n]:=n!`
 `end do;`
 `s[1]:=1;`
 `for n from 2 to 8 do`
 `s[n]:=s[n-1]+n end do;`
	A typical for-loop (`for` and `do` statement) used in general programming languages. It executes whatever between ``do'' and ``end do'' repeatedly for a counted number of times (``for n from i to j''). It hence can be used to work with sequences in much more general ways than what the command `seq` could.

- A link to the *SequenceDrill* maplet can be found on the course website:

    `http://www.math.sc.edu/calclab/142L-F10/labs/`

Related course material/Preparation

Calculus Text: §11.1 and §11.2. Maple Text: Sections §9.1 and §9.2. In addition, review the basic qualitative properties of logarithms, powers, exponentials, and so on. For example, exponentials grow faster (at ∞) than polynomials, factorials grow faster than exponentials, and so on.

Assignment

Complete activities on the back of this page and your lab instructor will give other assignment for each section.

Activities

1. For each of the following sequences, generate and plot the first 10 terms and determine if it diverges or converges to a limit.

   (1) $\{1 + (-1)^n\}_{n=1}^{\infty}$ 　　　　　(2) $\{(-1)^n \arctan(n)\}_{n=1}^{\infty}$ 　　(3) $\left\{\sqrt{n^2 + pn} - n\right\}_{n=1}^{\infty}$

   (4) $\left\{\dfrac{10^n}{n!}\right\}_{n=0}^{\infty}$ 　　　　(5) $\left\{n \sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ 　　(6) $\left\{\ln(\dfrac{1}{n})\right\}_{n=1}^{\infty}$

   (7) $\left\{\dfrac{3 + n^2 \sin(n)}{2 + n^2}\right\}_{n=1}^{\infty}$ 　(8) $\left\{\displaystyle\sum_{k=1}^{n} \dfrac{1}{k^2}\right\}_{n=1}^{\infty}$ 　　(9) $\left\{\displaystyle\sum_{k=1}^{n} \dfrac{1}{1 + (k/n)}\right\}_{n=1}^{\infty}$

   Note: *You may use the* SequenceDrill *maplet. However, it does not work well with sequences involving parameters such as (3), since it involves* `plot`. *We will work out some examples using explicit commands.*

2. A typical format for a recursively-defined sequence is $a_{n+1} = f(a_n)$, $n = 2, 3, \cdots$ with a_1 given explicitly first. <u>Under the assumptions</u> that (i) $\{a_n\}$ converges to L and (ii) $f(x)$ is continuous at L, we have that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ and therefore

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

   Thus, L must be a solution to $L = f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (numerically, graphically, or exactly).

   (a) (See Exercise 68 on page 685) Consider the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$, $n = 1, 2, 3, \cdots$. Use Maple to verify that it is a bounded monotone sequence and hence converges to a limit. Explain how a plot containing the graphs of $y = x$ and $y = \sqrt{2 + x}$ confirms this limit.

   (b) Consider the sequence $\{x_n\}$ produced by Newton's Method (§4.8) to approximate $\sqrt{2}$ as a zero of $f(x) = x^2 - 2$, where $x_1 = 1$, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$, $n = 1, 2, 3, \cdots$. Use Maple to verify that the limit is indeed $\sqrt{2}$.

Examples:

```
>restart;
>with(plots):
(8) in Activity 1:
>a8:= n->sum(1/k^2, k=1..n);
>seq(a8(n), n=1..10);
>p8:=evalf(seq([n,a8(n)], n=1..10)):
>plot([p8], style=point);
>limit(a8(n), n=infinity);
(a) in Activity 2:
>a[1]:=sqrt(2);
>for n from 1 to 9 do a[n+1]:=sqrt(2+a[n]); evalf(a[n+1]) end do;
>plot([x,sqrt(2+x)],x=-4..4);
>solve(x=sqrt(2+x),x);
```