TOTALLY REFLEXIVE MODULES AND ALMOST GORENSTEIN RINGS

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Abstract. This is a preliminary version.

Introduction

The following are general problems we would like to solve:

- Give necessary and sufficient conditions in certain family of ring for totally refexive modules to exist.
- If we know totally reflexive modules exist, find a construction to build one, and if the ring is not Gorenstein, to build infinitely many.

We provide partial answers to the first question for the class of almost Gorenstein rings, as defined in [HV].

1. The canonical module over almost Gorenstein rings

Two versions of the almost Gorenstein property are proposed in [HV]. In order to distinguish between them, we will refer to the second version as strongly almost Gorenstein.

(1.1) **Definition.** An artinian local ring $(R, \mathfrak{m})$ is *almost Gorenstein* if the inclusion $0 :_R (0 : I) \subseteq (I :_R \mathfrak{m})$ holds for every ideal $I \subset R$.

(1.2) **Definition.** An artinian local ring R is *strongly almost Gorenstein* if it has the property that $\omega_R^*(\omega_R) \supseteq \mathfrak{m}$, where ω_R denotes the canonical module of R, and $\omega_R^*(\omega_R) = \{y \in R \,|\, y = f(x)$ for $x \in \omega_R$ and some $f \in \omega_R^*\}$.

It is shown in [HV] that strongly almost Gorenstein implies almost Gorenstein. However, the two properties are not equivalent.

The class of Teter rings is a particular example of strongly almost Gorenstein rings

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(1.3) **Definition.** We say that R is a Teter ring if there exists a local artinian Gorenstein ring $(S, \mathfrak{m}_S)$ such that $R = S/(\delta)S$, where $(\delta)S = (0) :_S \mathfrak{m}_S$

For any artinian ring R, one may assume, by the Cohen structure theorem, that R is a quotient S/J where S is a Gorenstein artinian ring. If S is a Gorenstein ring, then $0 :_S (0 : I) = I$ for all ideals I in S. Therefore without loss of generality we may assume that $J = 0 : K$ for some ideal $K \subseteq S$. Assume that K is generated by $f_1, \ldots f_n$.

(1.4) **Lemma.** Let S be a Gorenstein artinian ring and let $K = (f_1, \ldots, f_n)$ be an ideal such that the ring $R = S/(0 :_S K)$ is almost Gorenstein. Then the following equality holds:

$$\mathfrak{m}_S = f_i :_S (f_1, \ldots, \hat{f}_i, \ldots f_n) + (f_1, \ldots, \hat{f}_i, \ldots f_n) :_S f_i.$$

Proof. Without loss of generality we may assume that $i = 1$. Let $I = (0 :_S f_1)$ and denote by $J = (0 :_S K)$. As $J \subseteq I$, then $J :_S (J :_S I) \subset I :_S \mathfrak{m}_S$. For the first term of the equality, we have the following equalities:

$$\begin{aligned} J :_S (J :_S I) &= (0 :_S K) :_S (J :_S I) \\ &= (0 :_S K(J :_S I)) \\ &= (0 :_S K[(0 :_S K) :_S I)]) \\ &= (0 :_S K(0 :_S KI)) \\ &= (0 :_S (0 :_S KI)) :_S K \\ &= KI :_S K \end{aligned}$$

For the rigth hand term of the equality:

$$I :_S \mathfrak{m}_S = (0 : f_1) :_S \mathfrak{m}_S = 0 :_S f_1\mathfrak{m}.$$

So that $KI :_S K \subseteq 0 :_S f_1\mathfrak{m}$. Using that S is Gorenstein, we have $(f_1\mathfrak{m}_S) = 0 :_S (0 :_S f_1\mathfrak{m}_S) \subseteq 0 :_S (KI :_S K)$. But

$$\begin{aligned} 0 :_S (KI :_S K) &= K(0 :_S KI) \\ &= K[(0 :_S I) :_S K] \\ &= K(f_1 :_S K) \end{aligned}$$

Putting all together we obtain that $f_1\mathfrak{m} \subseteq K(f_1 :_S K)$. In particular, for every element $x \in \mathfrak{m}$ we can write $xf_1 = \sum_{i=1}^{n} u_i f_i$, with $u_i \in (f_1 :_S K)$ and hence $(x - u_1)f_1 = \sum_{i=2}^{n} u_1 f_i$. This implies that $x - u_1 \in (f_2, \ldots, f_n) :_S f_1$ and finally $x \in (f_2, \ldots, f_n) :_S f_1 + f_1 :_S K =$

$(f_2, \dots, f_n) :_S (f_1) + (f_1) :_S (f_2, \dots, f_n)$. As x is a random element in the maximal ideal $\mathfrak{m}$, we have the thesis. □

(1.5) **Theorem.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local noetherian ring which is almost Gorenstein with canonical module ω_R. Assume that R is not Gorenstein and $R = S/J$, where S is an artinian Gorenstein ring, and $J :_S \mathfrak{m} \neq \mathfrak{m}J :_S \mathfrak{m}$. Then the residue field k is a direct summand of the second syzygy of the canonical module ω_R.

Proof. In the following, denote by y' the image in R of the element $y \in S$. Since S is Gorenstein, we may assume that $J = (0 :_S K)$, for some ideal $K = (f_1, \dots, f_n)$. The canonical module ω_R is given by $\mathrm{Hom}_S(R, S) = \mathrm{Hom}_S(S/(0 : K), S)$ which is isomorphic to $(0 : (0 : K)) = K$. Let

$$\cdots \longrightarrow R^p \xrightarrow{\partial_2} R^m \xrightarrow{\partial_1} R^n \xrightarrow{\partial_0} K \longrightarrow 0$$

be a minimal presentation of the canonical module. By Lemma 1.4, we can choose a set of minimal generators $x_1, \dots, x_e$ of the maximal ideal $\mathfrak{m}_S$, such that $x_i \in f_1 :_S (f_2, \dots f_n)$ or $x_i \in (f_2, \dots, f_n) : f_1$. In any case there is a relation $a_{1i}f_1 + a_{2i}f_2 + \cdots + a_{ni}f_n = 0$ in S, such that either $a_{1i} = x_i$ or $a_{2i} = x_i$. The column vectors $D'_i = (a'_{1i}, \dots, a'_{ni})$ are part of a minimal generating set for the first syzygy. After a choice of basis, $D'_1, \dots, D'_e$ are the first e columns of the matrix representing ∂_1, let $D'_{e+1}, \dots, D'_m$ be the other columns. For every $u' \in \operatorname{socle} R$ the element $(0, \dots, u', \dots, 0)$ is in the kernel of ∂_1. We claim that there exists a $u' \in \operatorname{socle} R$ such that $\mathbf{u}'_1 = (u', 0, \dots, 0)$ or $\mathbf{u}'_2 = (0, u', 0, \dots, 0)$ is a minimal generator for the second syzygy of E, if so then this shows that k is a direct summand of the second syzygy.

To prove the claim, denote by $B' = (b'_{ij})$ the matrix representing ∂_2. Let $u' \in \operatorname{socle} R$ and assume that $\mathbf{u}'_i$ is not part of a minimal set of generators of the second syzygy. This implies that

$$\mathbf{u}_i = c_1 \begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix} + \cdots + c_p \begin{pmatrix} b_{1p} \\ \vdots \\ b_{mp} \end{pmatrix} + JS^m$$

where c_i are elments of the maximal ideal $\mathfrak{m}_S$. Moreover we have

$$b_{1i}D_1 + \cdots + b_{mi}D_m \in JS^n$$

for all $i = 1, \dots, p$. This implies that $D\mathbf{u} = \sum c_j(b_{1j}D_1 + \cdots + b_{mj}D_m) \in \mathfrak{m}JS^n$. This implies that for all $u \in (J :_S \mathfrak{m})$ and for all $i = 1, \dots, e$ we have $ux_i \in \mathfrak{m}J$ or $(J :_S \mathfrak{m})\mathfrak{m} \subset \mathfrak{m}J$ which contradicts the assumtion $(J :_S \mathfrak{m}) \neq (\mathfrak{m}J :_S \mathfrak{m})$. □

2. Almost Gorenstein rings and totally reflexive modules

(2.1) **Lemma.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring with canonical module ω_R. If k is a direct summand of any syzygy of ω_R then there are no non-free totally reflexive modules.

Proof. Let X be a totally reflexive module, the equality $\operatorname{Ext}^1_R(X, M) = \operatorname{Ext}^{i+1}_R(X, \Omega^i_R M)$ holds for every R-module M. In particular for $M = \omega_R$ then $\operatorname{Ext}^{i+1}_R(X, \Omega^i_R \omega_R) = 0$ and $\operatorname{Ext}^{i+1}_R(X, \mathsf{k}) = 0$ if k is a direct summand of $\Omega^i_R(\omega_R)$. This shows that X has finite projective dimension and therefore it is free. □

(2.2) **Remark.** In [**?**] it is shown that if a local ring R can be written as a quotient S/J such that $\dim_{\mathsf{k}}(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m}) \geq 2$ then there are no totally reflexive modules. Theorem 1.5 shows that for almost Gorenstein rings the conclusion holds even in the case when $\dim_{\mathsf{k}}(J : \mathfrak{m})/(\mathfrak{m}J : \mathfrak{m}) \geq 1$.

(2.3) **Corollary.** Let R be a Teter ring, then every totally reflexive module is free.

Proof. Write $R = S/\delta$ where S is a Gorenstein artinian ring with socle equal to δ. The condition $(\delta :_S \mathfrak{m}) \neq (0 :_S \mathfrak{m}) = \delta$ holds, therefore one can apply Theorem 1.5 and Lemma 2.1, □

(2.4) **Example.** The ring $R = \mathsf{k}[x, y, z]/(x^2, y^2, z^2, yz)$ has totally reflexive modules which are not free. On the other hand, let $S = \mathsf{k}[x, y, z]/(x^2, y^2, z^2)$ and $J = (yz)S$ then $(J :_S \mathfrak{m}_S) = (\mathfrak{m}_S J ; \mathfrak{m}_s)$. The ring R has gorenstein colength 2.

The following proof is an adaptation from [HV]

(2.5) **Lemma.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay ring such that $\mathfrak{m}\operatorname{Ext}^1_R(M, R) = 0$ for all maximal Cohen-Macaulay module M. Then $R/\boldsymbol{x}$ is an Almost Gorenstein ring for all system of parameters $\boldsymbol{x}$.

Proof. Let I be any ideal of R containing the ideal generated by $\boldsymbol{x}$. We need to show that $\boldsymbol{x} :_R (\boldsymbol{x} :_R I) \subseteq I :_R \mathfrak{m}$. Assume that I is generated by $f_1, \ldots, f_n$ and consider the short exact sequence

$$0 \to \frac{R}{\boldsymbol{x} : I} \to \oplus \frac{R}{I} \to N \to 0,$$

where the first map is given by $\overline{u} \to (\overline{f_1 u}, \ldots, \overline{f_n u})$. Applying the functor $\operatorname{Hom}_R(\quad, R/\boldsymbol{x})$ to the short exact sequence we obtain:

$$0 \to \operatorname{Hom}_R(N, \frac{R}{\boldsymbol{x}}) \to \operatorname{Hom}_R(\oplus \frac{R}{I}, \frac{R}{\boldsymbol{x}}) \to \operatorname{Hom}_R(\frac{R}{\boldsymbol{x} : I}, \frac{R}{\boldsymbol{x}}) \to \operatorname{Ext}^1_R(N, \frac{R}{\boldsymbol{x}}).$$

The cokernel of the middle map is the cokernel of:

$$\oplus \frac{\boldsymbol{x} :_R I}{\boldsymbol{x}} \to \frac{\boldsymbol{x} :_R (\boldsymbol{x} :_R I)}{\boldsymbol{x}}$$

given by $(\overline{u}_1, \ldots, \overline{u}_n) \to \overline{f_1 u_1 + \cdots + f_n u_n}$. The cokernel is therefore isomorphic to $\frac{\boldsymbol{x}:_R(\boldsymbol{x}:_R I)}{I}$ and embeds in $\operatorname{Ext}^1_R(N, R/\boldsymbol{x})$. This last module is isomorphic to $\operatorname{Ext}^{d+1}_R(N, R)$ which is isomorphic to $\operatorname{Ext}^1_R(\Omega^d(N), R)$ and therefore annihaleted by $\mathfrak{m}$. This implies that $\mathfrak{m}\frac{\boldsymbol{x}:_R(\boldsymbol{x}:_R I)}{\boldsymbol{x}} = 0$ and therefore the thesis. □

(2.6) **Remark.** In [AGP], Theorem 3.1, the authors prove that if $(R, \mathfrak{m}, \mathsf{k})$ is a local ring and $\boldsymbol{y} = y_1, \ldots, y_d$ is a regular sequence in $\mathfrak{m}^2$ then $R/\boldsymbol{y}$ has a totally reflexive module.

(2.7) **Remark.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring. Let M be a finitely generated R-module. For every element x of the maximal ideal, denote by μ_x the multiplication by x. If for every x in a minimal set of generators of $\mathfrak{m}$ there exists a linear map ϕ_x such that the daigram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow \mu_x & \swarrow \phi_x & & & & & \\ 0 & \longrightarrow & \Omega^1(M) & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes, then $\mathfrak{m} \operatorname{Ext}^1_R(M, N) = 0$ for all modules N.

(2.8) **Example.** In this example we show that there exists an almost gorenstein ring that admits a totally reflexive module. The ring $R = \mathbb{C}[[x, y, z, u, v]]/(xz - y^2, xv - yu, yv - zy)$ is of finite Cohen Macaulay type and its only indecomposable maximal Cohen-Macaulay modules are R, the ideals $\alpha = (x, y)$, $\alpha^2 = (x^2, y^2, xy)$, $\beta = (x, y, u)$ and the R-module $\Omega^1_R(\beta)$. For a proof of this see for example [?]. In the following we show that the maximal ideal $\mathfrak{m}$ annihalets all the R-modules $\operatorname{Ext}^1_R(M, R)$ for M maximal Cohen-Macaulay. We first show that $\mathfrak{m} \operatorname{Ext}^1_R(\alpha, \Omega^1(\alpha)) = 0 = \mathfrak{m} \operatorname{Ext}^1_R(\beta, \Omega^1(\beta))$.

(1) For the ideal α, the first syzygy $\Omega^1_R(\alpha)$ is generated by

$$[-v, u], [-z, y], [-y, x].$$

The following list gives the maps of Remark 2.7

$$\phi_x = \begin{pmatrix} y & z-y \\ -x & -y+x \end{pmatrix} \qquad \phi_y = \begin{pmatrix} -y & 0 \\ x & 0 \end{pmatrix}$$
$$\phi_z = \begin{pmatrix} z & 0 \\ -y & 0 \end{pmatrix} \qquad \phi_v = \begin{pmatrix} v-y & -z \\ -u+x & y \end{pmatrix}$$
$$\phi_u = \begin{pmatrix} y & 0 \\ -u & 0 \end{pmatrix}$$

(2) For the ideal β, the first syzygy $\Omega^1_R(\beta)$ is generated by

$$[0,-v,z],[-v,u,0],[-v,0,y],[-u,0,x],[-z,y,0],[-y,x,0].$$

The following list gives the maps of Remark 2.7

$$\phi_x = \begin{pmatrix} 0 & -y & -u \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \qquad \phi_y = \begin{pmatrix} 0 & -z & -v \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix}$$
$$\phi_z = \begin{pmatrix} z & 0 & 0 \\ -y & 0 & -v \\ 0 & 0 & z \end{pmatrix} \qquad \phi_v = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ -y & -z & 0 \end{pmatrix}$$
$$\phi_u = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ -x & -y & 0 \end{pmatrix}$$

By [?], there exists a short exact sequence $0 \to R \to \alpha \oplus \alpha \oplus \beta \to \Omega^1_R(\beta) \to 0$. By applying the functor $\mathrm{Hom}_R(-,N)$ we can see that $\mathrm{Ext}^1_R(\Omega^1_R(\beta),N)$ is annihilated by the maximal ideal $\mathfrak{m}$ for every R-module N. All it is left to show is that $\mathfrak{m}\,\mathrm{Ext}^1_R(\alpha^2,R)=0$, which can be checked using Macaulay 2. The conclusion follows from Lemma 2.5 and Remark 2.6.

3. The monomial case

The main result of this section deals with artinian strongly almost Gorenstein rings which are obtained as quotients of polynomial rings by monomial ideals.

(3.1) **Theorem.** Let $S = \mathsf{k}[x_1,\ldots,x_d]/(x_1^{A_1},\ldots,x_d^{A_d})$, and let $f_1,\ldots,f_n$ be monomials in S such that

(1) f_i does not divide f_j for every $i \neq j$;
(2) x_i divides f_j for all $i=1,\ldots,d$ and for all $j=1,\ldots n$;
(3) $(x_1,\ldots,x_u) \subseteq \sum_{i=1}^n f_i :_S (f_1,\ldots,f_n)$.

then, one of the following holds:

(A) There exists an integer $i \in \{1, \dots, n\}$ and an integer $j \in \{1, \dots, u\}$ such that

$$\frac{f_i}{x_j} \in (f_1, \dots, f_n) : (x_1, \dots, x_u);$$

(B) There exists a partition $S_1 \cup \dots \cup S_n$ of $\{1, \dots, u\}$ such that for evey integer $i \in \{1, \dots, n\}$ one has $x_j f_i = 0$ for every $j \in \{1, \dots, u\} \setminus S_i$ and $x_j f_i \neq 0$ for all $j \in S_i$.

Note that if $R = S/0 :_S (f_1, \dots, f_n)$ is strongly almost Gorenstein, then (2) above holds by [HV]. We may assume without loss of generality that (1) and (3) also hold. Indeed, note that R does not change if we replace S by $S' = \mathsf{k}[x_1, \dots, x_d]/(x_1^{A_1+1}, \dots, x_d^{A_d})$, and $f_1, \dots, f_n$ by $f_1', \dots, f_n'$, where $f_i' = (x_1 \cdots x_d) f_i$.

Proof. Before we proceede with the proof, we establish some claims that we will use later. Write each $f_i = \Pi_{i=1}^n x_i^{N_{ji}}$.

Claim 1: If $x_i f_j \in (f_k)$, for some integers i, j, k then one of the following cases hold:

(i) $\begin{cases} N_{ji} = N_{ki} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i \end{cases}$

(ii) $N_{ji} = A_j - 1$

Moreover, the first case holds for just one index i.

Indeed, assume that the second possibility does not hold then $N_{ji} + 1 \geq N_{ki}$ and $N_{jl} \geq N_{kl}$ for every $l \neq i$. If $N_{ji} + 1 > N_{ki}$ then $N_{jl} \geq N_{kl}$ for every $l \in \{1, \dots, n\}$ which implies that f_k divides f_i, contradicting the hyphotesis. For the last statement, assume that there are two indeces i_1 and i_2 such that

$$\begin{cases} N_{ji_1} = N_{ki_1} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i_1 \end{cases}$$

and

$$\begin{cases} N_{ji_2} = N_{ki_2} - 1 \\ N_{jl} \geq N_{kl}, \quad \text{for every} \quad l \neq i_2 \end{cases}$$

then, $N_{ki_2} - 1 = N_{ji_2} \geq N_{ki_2}$ which is a contradiction.

Claim 2: If conclusion B holds then we may assume the following:

(i) each set S_i has cardinality bigger than 2;
(ii) for every $k \in S_i$ we have $x_k \in (f_i) : (f_1, \dots, f_n)$.

Indeed assume that there exist indeces i and k such that $S_i = \{x_k\}$, then case (A) holds as $\frac{f_i}{x_k} \in (f_i) : (f_1, \ldots, f_n)$. For (ii), let $k \in S_i$. Note that by hyphotesis (3) we have $x_k \in (f_j) : (f_1, \ldots, f_n)$ for some $j \neq i$. Then $x_k f_i \in (f_j)$. As we may assume (i), there exists an $l \in S_i$ such that $l \neq k$. For such an index we have $N_{il} \geq N_{jl}$. As the S_i for a partition of the variables $\{x_1, \ldots x_u\}$, $x_l \notin S_j$ and therefore $x_l f_j = 0$ which implies that $N_{jl} = A_j - 1$ and therefore $N_{il} = A_j - 1$. It follows that $x_l f_i = 0$ which is a contradiction as B holds and $x_k \in S_i$. We will assume (i) and (ii) everytime we assume that B holds.

The proof of the theorem goes by induction on the number of variables d, the case $d = 1$ being obviously true. Assume that the theorem holds for $d - 1$ variables. We now induct on the number of polynomials. For $n = 2, 3$ the theorem is settle by. Assume that the theorem holds in the case of $n - 1$ polynomials.

Claim 3: Assume that conclusion B holds for $f_1, \ldots, f_{n-1}$ with respect of a set of variables $x_1, \ldots, x_s$, with $s \leq u$. Then, one has $N_{nk} = N_{ik} - 1$ and $N_{nl} \geq N_{il}$, for all $l \neq k$, for at most just one i and k.

proof of Claim 3: Assume that conclusion (B) holds for $f_1, \ldots, f_{n-1}$. Therefore there is a partition

$$S'_1 \cup \cdots \cup S'_{n-1} = \{1, \ldots, s\} \tag{3.1.1}$$

such that $x_k f_i \neq 0$ for all $k \in S_i$ and $x_k f_i = 0$ for all $k \notin S_i$. By Claim 2, for every $k \in S_i$, $x_k \in (f_i) : (f_1, \ldots, f_{n-1})$. By hypothesis $x_k \in (f_j) : (f_1, \ldots, f_n)$, for some j and therefore $x_k \in (f_j) : (f_1, \ldots, f_{n-1})$. Since B holds and 3.1.1 is a partition, we have $j = i$. It follows that for every $k \in S_i$, $x_k f_n \in (f_i)$, and therefore the equations of Claim 1 hold. Note that one has $N_{nk} = N_{ik} - 1$ and $N_{nl} \geq N_{il}$, for all $l \neq k$, for just one i and k. For if $x_{k_1} f_n \in (f_{i_1})$ and $x_{k_2} f_n \in (f_{i_2})$ with $x_{k_1} f_n \neq 0 \neq x_{k_2} f_n$. If $i_1 = i_2$ then this follows from Claim 1. Assume that $i_1 \neq i_2$ and for example that $N_{nk_1} = N_{i_1 k_1} - 1$, and $N_{nl} \geq N_{i_2 k_2}$. Since (B) holds then $x_{k_1} f_{i_2} = 0$ which implies that $x_{k_1} f_n = 0$ and $N_{nk_1} = A_{k_1} - 1$. Finally, $N_{nk_1} + 1 = N_{i_1 k_1} = A_{k_1}$, which is absurd as every variable divides the monomials f_l.

Claim 4: For every $i \in \{1, \ldots, n\}$ the following holds

$$(x_1, \ldots, x_u) \nsubseteq \sum_{l \neq i} (f_l) : (f_1, \ldots, f_n).$$

Assume, by way of contradiction, that $(x_1, \ldots, x_u) \subseteq \sum_{i=1}^{n-2} f_i : (f_1, \ldots, f_{n-1})$ then conditions 1,2,3 of the theorem are satisfied and by the induction hyphothesis either A of B holds. If (A) holds for the monomials $f_1,\dots,f_{n-1}$ then it holds for $f_1,\dots,f_n$. If B holds then, by claim 3 with $s=u$, one has $x_kf_n=0$ for every k but one, say l. For such integer we have

$$\frac{f_n}{x_l}\in(f_1,\dots,f_n):(x_1,\dots,x_u).$$

Because of Claim 4, by renaming the variables we may assume that

$$x_1,\dots,x_s\notin(f_n):(f_1,\dots,f_{n-1},f_n) \tag{3.1.2}$$

$$x_{s+1},\dots,x_u\in(f_n):(f_1,\dots,f_{n-1},f_n) \tag{3.1.3}$$

We apply the induction hypothesis to the monomials $f_1,\dots,f_{n-1}$ with respect to the variables $x_1,\dots,x_s$.

Assume that conclusion B holds, by Claim 3, we have

$$x_kf_n=0,\qquad \text{for all but possibly one}\quad k\in\{1,\dots,s\}. \tag{3.1.4}$$

We now prove that $x_l\in(f_n):(f_1,\dots,f_{n-1})$ for all $s+1\le l\le u$, and 3.1.4 imply

$$x_lf_1=\dots x_lf_{n-1}=0,\quad \text{for all}\quad s+1\le l\le u. \tag{3.1.5}$$

Indeed, assume by way of contradiction that there exists a j such that $x_jf_i\neq0$ for some $j\ge s+1$ and some $i\le n-1$, then $0\neq x_jf_i\in(f_n)$. Equation 3.1.4 implies that $x_lf_i=0$ for all $l\le s$, but at most one. But this contradicts the fact that we are assuming conclusion B and that the set S_i' has cardinality bigger then two and that for every $j\in S_i'$, $x_jf_i\neq0$.

Equations 3.1.4 implies two possibilities:

(1) there exists an integer k, $1\le k\le s$, such that $x_kf_n\neq0$. If this is the case then we have $x_lf_n=0$. For if $x_kf_n\in(f_i)$ for $k\in S_i$, with $i\le n-1$ and $x_lx_kf_n\in(x_lf_i)=0$, by 3.1.5. In this case conclusion A holds as we have

$$\frac{f_n}{x_k}\in(f_1,\dots,f_n):(x_1,\dots,x_u).$$

(2) x_kf_n0 for all $1\le k\le s$. In this case conclusion B holds with th partition $S_i=S_i'$ and $S_n=\{1,\dots,s\}$.

Assume conclusion A holds for $\{f_1,\dots,f_{n-1}\}$ with respect to the variables $\{x_1,\dots x_s\}$. Withou loss of generality, we may assume that

$$\frac{f_1}{x_1}\in(f_1,f_2,\dots f_{n-1}).$$

If we had

$$x_l\frac{f_1}{x_1} \in (f_n), \quad \text{for every} \quad s+1 \le l \le u, \tag{3.1.6}$$

then conclusion A would hold for the original set of monomials $\{f_1, \dots, f_n\}$. As $x_l f_1 \in (f_n)$ for all $s+1 \le l \le u$, f $x_l f_1 = 0$ for all $s+1 \le l \le u$, or $N_{11} > N_{n1}$ then equations 3.1.6 hold. Without loss of generality we may assume that

$$N_{11} \le N_{n1} \tag{3.1.7}$$

and $x_l f_1 \ne 0$ for some $s+1 \le l \le u$. By Claim 1, there exists just one value of l, say $s+1$ such that $x_l f_1 \ne 0$, as $x_l f_1 \in (f_n)$. So we may assume

$$x_{s+1} f_1 \ne 0, \text{ with } N_{11} = N_{n1} \text{ and } x_l f_1 = 0, \text{ for all } s+2 \le l \le u \tag{3.1.8}$$

Claim 5: The following holds:

$$x_2 f_1 = \dots x_s f_1 = 0 \tag{3.1.9}$$

If, say, $x_2 f_1 \ne 0$, then $0 \ne x_2\frac{f_1}{x_1} \in (f_i)$ for some $i \le n-1$, which implies that $N_{11} > N_{i1}$. As, by assumptions, $x_{s+1} f_i \in (f_n)$ then we obtain the two following possibilities:

(1) either $x_{s+1} f_i = 0$, which implies $x_{s+1} f_1 = 0$, contradiction 3.1.8; or
(2) $N_{i1} \ge N_{n1}$, which implies $N_{11} > N_{n1}$, contradictiong 3.1.7.

This proves claim 5.

Because of Claim 4, there exists an index j, such that $1 \le j \le s$ and

$$x_j \in (f_1) : (f_2, \dots, f_n) \tag{3.1.10}$$

We may assume that

$$x_1 f_1 \ne 0, \quad \text{and therefore } x_1 f_n \ne 0 \tag{3.1.11}$$

otherwise, by 3.1.8 and 3.1.9, $x_l f_1 = 0$ for all $l \in \{1, \dots, s, s+2, \dots, u\}$. It follows that condition A holds:

$$\frac{f_1}{x_{s+1}} \in (f_1, \dots, f_n) : (x_1, \dots, x_u).$$

The following cases finish the proof of the theorem.

(1) Assume $j = 1$. Since $x_1 f_n \in (f_1)$ and $N_{11} = N_{n1}$, by 3.1.8, then $x_1 f_n = 0$ contradicting 3.1.11.
(2) Assume $j \ge 2$. We may assume $j = 2$. By 3.1.8 and 3.1.9 we have $x_l f_1 = 0$ for all $l \ne 1, s+1$. We may assume that $x_1 f_1 \ne 0$, by 3.1.11.

(a) Assume that $x_2f_n \neq 0$. Since $x_1 \in (f_i) : (f_1, \dots, f_n)$ for some $i \in \{1, \dots, n-1\}$. As $0 \neq x_2f_n \in (f_1)$ and $x_1f_n \neq 0$, by Claim (1) we need to have $2 \leq i \leq n-1$. For such an i, we claim that

$$\frac{f_i}{x_1} \in (f_1, \dots, f_n) : (x_1, \dots, x_u). \tag{3.1.12}$$

First notice that $N_{i1} = N_{n1}+1 = N_{11}+1$, since $0 \neq x_1f_n \in (f_i)$ and by 3.1.7. Moreover, as $x_2f_n \neq 0$, by multiplying x_1f_n by x_2 we obtain that $0 \neq x_2f_i \in (f_1)$. If $x_lf_i \neq 0$ for $l \neq 1, 2, s+1$, then $x_lf_1 \neq 0$, contradicting 3.1.7 and 3.1.8. As $x_{s+1} \in f_n : (f_1, \dots, f_n)$, we obtain $x_{s+1}f_i \in (f_n)$ and since $N_{i1}N_{n1}+1$ also $x_{s+1}\frac{f_i}{x_1} \in (f_n)$.

(b) Assume that $x_2f_n = 0$. If $x_2f_i = 0$, for all $i \in \{1, \dots, u\}$ then we can ignore the variable x_2 and the conclusion of the theorem will hold by induction. So we may assum that there is a $t \neq 1, n$ such that $x_2f_t \neq 0$ and $x_2f_t \in (f_1)$. Therefore $N_{12} = N_{t2}+1$. As $x_lf_t \in f_n$ for every $s+1 \leq l \leq u$, if $x_lf_t \neq 0$ then $A_2 - 1 = N_{n2} \leq N_{t2}$ which implies that $N_{12} = A_2$ which is a contradiction as all the variables divide all the monomials. Therefore we have that $x_lf_t = 0$ for all $s+1 \leq t \leq u$. Also, as $0 \neq x_2f_t \in (f_1)$, we have $N_{tk} \geq N_{1k}$. As $x_lf_1 = 0$ for all $l \neq 1, s+1$, it follows that $x_lf_t = 0$ for all $l \neq 1, 2$. If also $x_1f_t = 0$ then conclusion A holds as

$$\frac{f_t}{x_2} \in (f_1, \dots, f_n) : (x_1, \dots, x_u).$$

Assume that $x_1f_t \neq 0$. Since $x_1 \in (f_i) : (f_1, \dots, f_n)$ for some i. As $1 < s$, $i \neq n$. We claim that

$$\frac{f_i}{x_1} \in (f_1, \dots, f_n) : (x_1, \dots, x_u).$$

AS $0 \neq x_1f_t \in (f_i)$, we have $N_{il} \geq N_{tl}$ for all $l \neq 1$. As $x_2f_t \neq 0$ this implies that $x_2f_i \neq 0$. As $x_2f_i \in (f_1)$, since $x_lf_1 = 0$ for $l \neq 1, s+1$, we obtain that $x_lf_i = 0$ for $l \neq 1, 2, s+1$. To prove the claim, it is therefore enough to prove that $\frac{f_i}{x_1}x_2 \in (f_1)$ and $\frac{f_i}{x_1}x_{s+1} \in (f_n)$. As $0 \neq x_1f_1 \in (f_i)$ we obtain $N_{i1} = N_{11}+1 = N_{n1}+1$, where the last equality follows from 3.1.8. This together with the fact that $x_2f_i \in (f_1)$ and $x_{s+1}f_i \in (f_n)$ concludes the claim.

□

(3.2) **Corollary.** Let $S = \mathsf{k}[x_1, \ldots, x_d]/(x_1^{A_1}, \ldots, x_d^{A_d})$, and let $f_1, \ldots, f_n$ be monomials in S such that $R = S/0 :_S (f_1, \ldots, f_n)$ is strongly almost Gorenstein. Then R does not admit non-free totally reflexive modules.

Proof. As noted immediately after the statement of Theorem **??**, the strongly amost Gorenstein assumption means that we can apply Theorem **??**. If (A) holds, then Theorem 1.5 can be applied to show that k is a direct summand of the second syzygy of ω_R, and the conclusion follows from Lemma 2.1.

If (B) holds, it is easy to check that k is a direct summand of the first syzygy of ω_R, and again the conclusion follows from Lemma 2.1. Indeed, note that

$$(x_1^{A_1}, x_2^{A_2}, \ldots, x_d^{A_d}) : (f_1, \ldots, f_n) = (x_1^{A_1}, x_2^{A_2}, \ldots, x_d^{A_d}) +$$
$$(x_j x_{j'} \mid x_j \text{ and } x_{j'} \text{ do not belong to the same } S_i).$$

The relations on the generators $f_1, \ldots, f_n$ of ω_R are $x_j f_i = 0$ for $j \notin S_i$, and $(\Pi_{j \in S_i} x_j^{A_j - 1}) f_i - (\Pi_{j \in S_{i'}} x_j^{A_j - 1}) f_{i'} = 0$. Note that the latter relations are killed by the maximal ideal, thus each of them generates a copy of k which splits off the first syzygy. ☐

References

[AGP] L. Avramov, V. Gasharov, and I. Peeva, *Complete intersection dimension*, Inst. Hautes ÂťEtudes Sci. Publ. Math. **86** (1997), 67–114

[HJS] M. Hughes, D. Jorgensen, and L. M. Şega, *On acyclic complexes of free modules*, Math. Scand., to appear.

[HV] C. Huneke and A. Vraciu, *Rings which are almost Gorenstein*, Pacific J. Math. **225** (2006), no. 1, 85–102.