Math 172 Fall 2012 Handout 8 September 18

The Euler method

The Euler method is used to find numerical approximations for solutions to differential equations. The main idea is to convert the differential equation $\frac{dP}{dt} = ...$ into a difference equation $\Delta\mathcal{P} = ...$ and then find numerical values using the calculator (enter the sequence as a recursive equation, the way we've done earlier).

In order to do this conversion, recall that the derivative $\frac{dP}{dt}$ is the limit of fractions of the form $\frac{\Delta P}{\Delta t}$ where Δt is a (specified) time interval. Thus, $\Delta\mathcal{P}$ can be approximated by $\Delta t\frac{dP}{dt}$. A smaller choice of the time interval Δt will lead to an approximation that is closer to the actual value (but will require more steps for the calculation to be completed).

This numerical approximation can be performed for any differential equation. We illustrate the method for the logistic equation

$$\frac{dA}{dt} = 0.0003A(200 - A)$$

with initial value $A(0) = 50$.

We will apply Euler's method to estimate $A(12)$. We will use a time interval $\Delta t = 4$. This means that the time interval from 0 to 12 is subdivided into three intervals of length 4 each. When the equation is converted to a discrete equation it will take the form

$$\Delta\mathcal{A} = \Delta t \times 0.0003\mathcal{A}(200 - \mathcal{A}) = 0.0012\mathcal{A}(200 - \mathcal{A})$$

We need to keep in mind that a unit of time for this discrete equation corresponds to an interval of time of length 4 for the original equation. Thus, $A(12)$ is the original equation corresponds to $\mathcal{A}(3)$ in the discrete version of the equation.

In order to enter the sequence $\mathcal{A}(n)$ in the calculator, we need to put the equation in the recursive equation form:

$$\mathcal{A}(n) = \mathcal{A}(n-1) + 0.0012\mathcal{A}(n-1)\,(200 - \mathcal{A}(n-1))$$

(use the letter "u" instead of $\mathcal{A}$ when you enter it in the calculator). We get $\mathcal{A}(3) = 79.83$, thus $A(12) \cong 79.83$ (recall that $\mathcal{A}(3)$ is an approximation for $A(12)$).

Repeating the Euler method with a time interval $\Delta t = 1$, it will now take 12 steps in order to reach $A(12)$. The difference equation now is

$$\Delta\mathcal{A} = \Delta t \times 0.0003\mathcal{A}(200 - \mathcal{A}) = 0.0003\mathcal{A}(200 - \mathcal{A})$$

In recursive form, we have

$$\mathcal{A}(n) = \mathcal{A}(n-1) + 0.0003\mathcal{A}(n-1)\,(200 - \mathcal{A}(n-1))$$

On the table of values obtained on the calculator we find $\mathcal{A}(12) = 80.93$, so $A(12) \cong 80.93$. Compared with the previous approximation of 79.83, this new approximation is closer to the actual value, since a discrete process with small time intervals is a better approximation of a continuous process.