Math 172 Fall 2012 Handout 5 September 11

A different method for solving affine equations

We wish to solve an affine equation of the form

$$\frac{dP}{dt} = rP - c \quad (\text{or} \frac{dP}{dt} = c - rP \quad \text{or} \Delta P = rP - c \quad \text{or} \Delta P = c - rP)$$

First we find the equilibrium value $P_{equil} = \dfrac{c}{r}$. Next we introduce a new function: $Q = P - P_{equil}$. What differential or difference equation is satisfied by Q?

Note that $\dfrac{dQ}{dt} = \dfrac{dP}{dt} = rP - c$. In order to express the right hand side of this equation in terms of Q, we replace P by $Q + P_{equil}$ so the equation becomes

$$\frac{dQ}{dt} = r(Q + P_{equil}) - c = rQ + r\frac{c}{r} - c = rQ$$

Thus Q satisfies the equation $\dfrac{dQ}{dt} = rQ$; this is an exponential model equation, and the solution is $Q(t) = Q_0 e^{rt}$ which allows us to obtain $P(t) = Q(t) + P_{equil} = Q_0 e^{rt} + \frac{c}{r}$. The last step is to find the value of Q_0 if P_0 is given, and that is $Q_0 = P_0 - \frac{c}{r}$, so the final answer is

$$P(t) = (P_0 - \frac{c}{r})e^{rt} + \frac{c}{r}$$

Note: If we want to solve the equation $\dfrac{dP}{dt} = c - rP$ instead, then the equation for Q will be $\dfrac{dQ}{dt} = -rQ$, so we will get $Q(t) = Q_0 e^{-rt}$, and the final answer will be

$$P(t) = (P_0 - \frac{c}{r})e^{-rt} + \frac{c}{r}$$

If we solve a difference equation $\Delta P = rP - c$ or $\Delta P = c - rP$ using this procedure, then e^{rt} should be replaced by $(1 + r)^t$ and e^{-rt} should be replaced by $(1 - r)^t$.