Review for Exam 3

1. If \( f : G_1 \to G_2 \) is a group isomorphism, prove that \( f^{-1} : G_2 \to G_1 \) is also an isomorphism.

2. If \( f : G_1 \to G_2 \) is a group isomorphism, and \( x \in G \), prove that \( o(x) = o(f(x)) \).

3. If \( f : G_1 \to G_2, g : G_2 \to G_3 \) are group isomorphisms, prove that \( g \circ f : G_1 \to G_3 \) is also a group isomorphism.

4. Prove that every subgroup of a cyclic group is cyclic.

5. Let \( a, n \) be integers and let \( d = \gcd(a, n) \). Prove that \( < [a]_n > = < [d]_n > \).

6. Let \( H_1 = a\mathbb{Z}, H_2 = b\mathbb{Z} \) be subgroups of \( \mathbb{Z} \), where \( a, b \) are integers. Prove that \( H_1 \cap H_2 = \text{lcm}(a, b)\mathbb{Z} \) and \( H_1 + H_2 = \gcd(a, b)\mathbb{Z} \).

7. State Cayley’s theorem. Assume that there is a function \( F : G \to S_n \) which is one-to-one and a group homomorphism. Explain how to use this to prove Cayley’s theorem.

8. Let \( G \) be a group of order \( n \). Explain how the function \( F : G \to S_n \) that was used in the proof of Cayley’s theorem is defined.

9. Let \( A_n \) be the set of all even permutations in \( S_n \). Prove that \( A_n \) is a subgroup of \( S_n \).

10. Prove that \( |A_n| = n!/2 \).

11. List all the elements of the dihedral group \( D_4 \) in each of the following way:
   a. Geometrically, as rotations and reflections.
   b. As permutations in \( S_4 \).
   c. In terms of generators and relations.

12. Find the center of the group \( D_4 \).

13. Let \( G = D_4 \), the group of symmetries of a square. Label the vertices of the square 1, 2, 3, 4 and consider the elements \( a=\text{rotation of angle } \pi/4 \) and \( b=\text{reflection about the diagonal } \overline{13} \). Describe the element \( ab \) as a geometric function (rotation or reflection - specify the angle for a rotation or the axis for a reflection).