The Mathematics of Game Shows

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These are the course notes for a class on The Mathematics of Game Shows which I taught at the University of South Carolina (through their Honors College) in Fall 2016, and again in Spring 2018. They are in the middle of revisions, being made as I teach the class a second time.

You can find the syllabus here:

`http://people.math.sc.edu/thornef/schc212/index.html`

These notes should be considered a first draft.

I welcome feedback from anyone who reads this (please e-mail me at `thorne[at]math.sc.edu`).

The notes contain clickable internet links to clips from various game shows, hosted on the video sharing site Youtube (`www.youtube.com`). These materials are (presumably) all copyrighted, and as such they are subject to deletion (indeed as of this writing the first Deal or No Deal link is already dead). I have no control over this. Sorry! If you encounter a dead link I recommend searching Youtube for similar videos. The Price Is Right videos in particular appear to be ubiquitous.

I would like to thank Bill Butterworth, Paul Dreyer, and **all** of my students for helpful feedback. I hope you enjoy reading these notes as much as I enjoyed writing them!

Contents

1 Introduction

To begin, let's watch some game show clips and investigate the math behind them.

Here is a clip from the game show **Deal or No Deal**:

Link: Deal Or No Deal – Full Episode

(If you are reading this on a computer with an internet connection, clicking on any line labeled 'Link' should bring up a video on a web browser.)

Game Description (Deal or No Deal): A contestant is presented with 26 briefcases, each of which contains some amount of money from \$0.01 to \$1,000,000; the amounts total \$3,418,416.01, and average \$131477.53. The highest prizes are \$500,000, \$750,000, and \$1,000,000.

The contestant chooses one briefcase and sets it aside. That is the briefcase she is playing for. Then, one at a time, she is given the opportunity the opportunity to open other briefcases and see what they contain. This narrows down the possibilities for the selected briefcase.

Periodically, the 'bank' offers to buy the contestant out, and proposes a 'deal': a fixed amount of money to quit playing. The contestant either accepts one of these offers, or keeps saying 'no deal' and (after opening all the other briefcases) wins the money in her original briefcase.

The **expected value** of a game is the average amount of money you expect to win. (We'll have much more to say about this.) So, at the beginning of the game, the expected value of the game is \$131,477.53, presuming the contestant rejects all the deals. In theory, that means that the contestant should be equally happy to play the game or to receive \$131,477.53. (Of course, this may not be true in practice.)

Now, consider this clip after the contestant has chosen six of the briefcases. Losing the \$500,000 was painful, but the others all had small amounts. After six eliminations, the total amount of prize money remaining is \$2,887,961.01, and the average is \$144,398.05 – higher than it was before. The banker offers him \$40,000 to stop playing. Since that is much lower than his expected value, understandably he refuses the offer and continues to play.

We now turn to the first game from this clip of The Price Is Right:

Link: The Price Is Right - Full Episode

Game Description (Contestants' Row - The Price Is Right): Four contestants are shown an item up for bid. In order, each guesses its price (in whole dollars). You can't use a guess that a previous contestant used. The winner is the contestant who bids the closest to the actual price without going over.

In the clip, the contestants are shown some scuba equipment, and they bid 750, 875, 500, and 900 in that order. The actual price is \$994, and the fourth contestant wins. What can we say about the contestants' strategy?

Who bid wisely? We begin by describing the results of the bidding. Let n be the price of the scuba gear.

- The first contestant wins if $750 \leq n \leq 874$.
- The second contestant wins if $875 \leq n \leq 899$.
- The third contestant wins if $500 \leq n \leq 749$.
- The fourth contestant wins if $900 \leq n$.
- If $n < 500$, then the bids are all cleared and the contestants start over.

We can see who did well before we learn how much the scuba gear costs. Clearly, the fourth contestant did well. If the gear is worth anything more than \$900 (which is plausible), then she wins. The third contestant also did well: he is left with a large range of winning prices – 250 of them to be precise. The second contestant didn't fare well at all: although his bid was close to the actual price, he is left with a very small winning range. This is typical for this game: it is a big disadvantage to go early.

The next question to ask is: could any of the contestants have done better?

We begin with the fourth contestant. Here the answer is *yes*, and her strategy is **dominated** by a bid of \$876, which would win whenever $900 \leq n$, and in addition when $876 \leq n \leq 899$. In other words: *a bid of \$876 would win every time a bid of \$900 would, but not vice versa.* Therefore it is *always* better to instead bid \$876.

Taking this analysis further, we see that there are exactly four bids that make sense: 876, 751, 501, or 1. Note that each of these bids, except for the one-dollar bid, screws over one of her competitors, and this is not an accident: Contestant's Row is a **zero-sum game** – if someone else wins, you lose. If you win, everyone else loses.

The analysis gets much more subtle if we look at the *third* contestant's options. **Assume that the fourth contestant will play optimally** (an assumption which is very often not true in practice). Suppose, for example, that the third contestant believes that the scuba gear costs around \$1000. The previous bids were \$750 and \$875. Should he follow the same reasoning and bid \$876? Maybe, but this exposes him to a devastating bid of \$877.

There is much more to say here, but we go on to a different example.

Game Description (Jeopardy, Final Round): Three contestants start with a variable amount of money (which they earned in the previous two rounds). They are shown a category, and are asked how much they wish to wager on the final round. The contestants make their wagers privately and independently.

After they make their wagers, the contestants are asked a trivia question. Anyone answering correctly gains the amount of their wager; anyone answering incorrectly loses it.

Link: Final Jeopardy – Shakespeare

Perhaps here an English class would be more useful than a math class! This game is difficult to analyze; unlike our two previous examples, the players play *simultaneously* rather than *sequentially*.

In this clip, the contestants start off with \$9,400, \$23,000, and \$11,200 respectively. It transpires that nobody knew who said that *the funeral baked meats did coldly furnish forth the marriage tables*. (Richard III? Really? When in doubt, guess Hamlet.) The contestants bid respectively \$1801, \$215, and \$7601.

We will save further analysis for later, but we will make one note now: the second contestant can obviously win. If his bid is less than \$600, then even if his guess is wrong he will end up with more than \$22,400.

In the meantime, imagine that the contestants started with \$6,000, \$8,000, and \$10,000. Then the correct strategy becomes harder to determine.

2 Probability

2.1 Sample Spaces and Events

At the foundation of any discussion of game show strategies is a discussion of *probability*. You have already seen this informally, and we will work with this notion somewhat more formally.

Definition 1 (Sample spaces and events): A **sample space** is the set of all possible outcomes of a some process. An **event** is any subset of the sample space.

Example 2: You roll a die. The sample space consists of all numbers between one and six.

Using formal mathematical notation, we can write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We can use the notation $\{\ldots\}$ to describe a set and we simply list the elements in it.

Let E be the *event* that you roll an even number. Then we can write

$$E = \{2, 4, 6\}.$$

Alternatively, we can write

$$E = \{x \in S \ : \ x \text{ is even}\}.$$

Both of these are correct.

Example 3: You choose at random a card from a poker deck. The sample space is the set of all 52 cards in the deck. We could write it

$$\begin{aligned} S = \{&A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, 9\clubsuit, 8\clubsuit, 7\clubsuit, 6\clubsuit, 5\clubsuit, 4\clubsuit, 3\clubsuit, 2\clubsuit, \\ &A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit, 9\diamondsuit, 8\diamondsuit, 7\diamondsuit, 6\diamondsuit, 5\diamondsuit, 4\diamondsuit, 3\diamondsuit, 2\diamondsuit, \\ &A\heartsuit, K\heartsuit, Q\heartsuit, J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit, 6\heartsuit, 5\heartsuit, 4\heartsuit, 3\heartsuit, 2\heartsuit, \\ &A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit, 9\spadesuit, 8\spadesuit, 7\spadesuit, 6\spadesuit, 5\spadesuit, 4\spadesuit, 3\spadesuit, 2\spadesuit\} \end{aligned}$$

but writing all of that out is annoying. An English description is probably better.

Example 4: You choose two cards at random from a poker deck. Then the sample space is the set of all pairs of cards in the deck. For example, $A\spadesuit A\heartsuit$ and $7\clubsuit 2\diamondsuit$ are elements of this sample space,

This is definitely too long to write out every element, so here an English description is probably better. (There are exactly 1,326 elements in this sample space.) Some events are easier to describe – for example, the event that you get a pair of aces can be written

$$E = \{A\spadesuit A\heartsuit, A\spadesuit A\diamondsuit, A\spadesuit A\clubsuit, A\heartsuit A\diamondsuit, A\heartsuit A\clubsuit, A\clubsuit A\diamondsuit\}$$

and has six elements. If you are playing Texas Hold'em, your odds of being dealt a pair of aces is exactly $\frac{6}{1326} = \frac{1}{221}$, or a little under half a percent.

Let's look at a simple example from the Price Is Right – the game of **Squeeze Play**:

Link: The Price Is Right - Squeeze Play

Game Description (Squeeze Play (The Price Is Right)): You are shown a prize, and a five- or six-digit number. The price of the prize is this number with one of the digits removed, other than the first or the last.

The contestant is asked to remove one digit. If the remaining number is the correct price, the contestant wins the prize.

In this clip the contestant is shown the number 114032. Can we describe the game in terms of a sample space?

It is imporrtant to recognize that **this question is not precisely defined. Your answer will depend on your interpretation of the question!** This is probably very much *not* what you are used to from a math class.

Here's one possible interpretation. Either the contestant wins or loses, so we can describe the sample space as

$$S = \{\text{you win}, \text{you lose}\}.$$

Logically there is nothing wrong with this. But it doesn't tell us very much about the structure of the game, does it?

Here is an answer I like better. We write

$$S = \{14032, 11032, 11432, 11402\},$$

where we've written 14032 as shorthand for 'the price of the prize is 14032'.

Another correct answer is

$$S = \{2, 3, 4, 5\},$$

where here 2 is shorthand for 'the price of the prize has the second digit removed.'

Still another correct answer is

$$S = \{1, 4, 0, 3\},$$

where here 1 is shorthand for 'the price of the prize has the 1 removed.'

All of these answers make sense, and all of them require an accompanying explanation to understand what they mean.

The contestant chooses to have the 0 removed. So the event that the contestant wins can be described as $E = \{11432\}$, $E = \{4\}$, or $E = \{0\}$, depending on which way you wrote the sample space. (Don't mix and match! Once you choose how to write your sample space, you need to describe your events in the same way.) If all the possibilities are equally likely, the contestant has a one in four chance of winning.

The contest guesses correctly and is on his way to Patagonia!

Definition 5 ($N(S)$): If S is any set (for example a sample space or an event), write $N(S)$ for the number of elements in it.

In this course we will always assume this number is *finite*.

Definition 6 (Probability): Suppose S is a sample space, **in which we assume that all outcomes are equally likely**.

For each event E in S, the **probability of E, denoted** $P(E)$, is

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 7: You roll a die, so $S = \{1, 2, 3, 4, 5, 6\}$.

1. Let E be the event that you roll a 4, i.e., $E = \{4\}$. Then $P(E) = \frac{1}{6}$.
2. Let E be the event that you roll an odd number, i.e., $E = \{1, 3, 5\}$. Then $P(E) = \frac{3}{6} = \frac{1}{2}$.

Example 8: You draw one card from a deck, with S as before.

1. Let E be the event that you draw a spade. Then $N(E) = 13$ and $P(E) = \frac{13}{52} = \frac{1}{4}$.
2. Let E be the event that you draw an ace. Then $N(E) = 4$ and $P(E) = \frac{4}{52} = \frac{1}{13}$.
3. Let E be the event that you draw an ace or a spade. What is $N(E)$? There are thirteen spades in the deck, and there are three aces which are not spades. Don't double count the ace of spades!

   So $N(E) = 13 + 3 = 16$ and $P(E) = \frac{16}{52} = \frac{4}{13}$.

Example 9: In a game of Texas Hold'em, you are dealt two cards at random in first position. You decide to raise if you are dealt a pair of sixes or higher, ace-king, or ace-queen, and to fold otherwise.

The sample space has 1326 elements in it. The event of two-card hands which you are willing to raise has 86 elements in it. (If you like, write them all out. Later we will discuss how this number can be computed more efficiently!)

Since all two card hands are equally likely, the probability that you raise is $\frac{86}{1326}$, or around one in fifteen.

Now, here is an important example:

Warning Example 10: You roll two dice and sum the totals. What is the probability that you roll a 7?

The result can be anywhere from 2 to 12, so we have

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and $E = \{7\}$. Therefore, we might be led to conclude that $P(E) = \frac{N(E)}{N(S)} = \frac{1}{11}$.

Here is another solution. We can roll anything from 1 to 6 on the first die, and the same for the second die, so we have

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,\\ 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}.$$

We list all the possibilities that add to 7:

$$E = \{16, 25, 34, 43, 52, 61\}$$

And so $P(E) = \frac{6}{36} = \frac{1}{6}$.

We solved this problem two different ways and got two different answers. **This illustrates the importance of our assumption that every outcome in a sample space will be equally likely**. This might or not be true in any particular situation. And one can't tell just from knowing what E and S are – one has to understand the actual situation that they are modelling.

We know that a die (if it is equally weighted) is equally likely to come up 1, 2, 3, 4, 5, or 6. So we can see that, according to our second interpretation, all the possibilities are still equally likely because all combinations are explicitly listed. But there is no reason why all the sums should be equally likely.

For example, consider the trip to Patagonia. If we assume that all outcomes are equally likely, the contestant's guess has a 1 in 4 chance of winning. But the contestant correctly guessed that over \$14,000 was implausibly expensive, and around \$11,000 was more reasonable.

Often, all events are *approximately* equally likely, and considering them to be exactly equally likely is a useful simplifying assumption.

We now take up the game **Rat Race** from The Price Is Right. (We will return to this example again later.)

Link: The Price Is Right - Rat Race

Game Description (Rat Race (The Price Is Right)): The game is played for three prizes: a small prize, a medium prize, and a car.

There is a track with five wind-up rats (pink, yellow, blue, orange, and green). They will be set off on a race, where they will finish in (presumably) random order.

The contestant has the opportunity to pick up to three of the rats: she guesses the price of three small items, and chooses one rat for each successful attempt.

After the rats race, she wins prizes if one or more of her rats finish in the top three. If she picked the third place rat, she wins the small prize; if she picked the second place rat, she wins the medium prize; if he picked the first place rat, she wins the car. (Note that it is possible to win two or even all three prizes.)

Note that except for knowing the prices of the small items, there is no strategy. The rats are (we presume) equally likely to finish in any order.

In this example, the contestant correctly prices two of the items and picks the pink and orange rats.

Problem. *Compute the probability that she wins the car.*

Solution 1. Here's the painful solution: describe all possible orderings in which the rats could finish. We can describe the sample space as

$$S = \{POB, POR, POG, PBR, PBG, PRG, \dots, \dots\}$$

where the letters indicate the ordering of the first three rats to finish. Any such ordering is equally likely. The sample space has sixty elements, and if you list them all you will see that exactly twenty-four of them start with P or G. So the probability is $\frac{24}{60} = \frac{2}{5}$.

Solution 2. Do you see the easier solution? To answer the problem we were asked, we only care about the **first** rat. So let's ignore the second and third finishers, and write the sample space as

$$S = \{P, O, B, R, G\}.$$

The event that she wins is

$$E = \{P, G\},$$

and so $P(E) = \frac{N(E)}{N(S)} = \frac{2}{5}$.

Solution 3 (Wrong). Here's another possible solution, which turns out to be wrong. It doesn't model the problem well, and it's very instructive to understand why.

As the sample space, take all combinations of one rat and which order it finishes in:

$$\begin{aligned} S = \{&\text{Pink rat finishes first,} \\ &\text{Pink rat finishes second,} \\ &\text{Pink rat finishes third,} \\ &\text{Pink rat finishes fourth,} \\ &\text{Pink rat finishes fifth,} \\ &\text{Yellow rat finishes first,} \\ &\text{etc.}\} \end{aligned}$$

This sample space indeed lists a lot of different things that could happen. But how would you describe the event that the contestant wins? If the pink or orange rat finishes first, certainly she wins. But what if the yellow rat finishes third? Then maybe she wins, maybe she loses. There are several problems with this sample space:

- The events are not mutually exclusive. It can happen that **both** the pink rat finishes second, **and** the yellow rat finishes first. A sample space should be described so that **exactly one of the outcomes will occur**.

  Of course, a meteor could strike the television studio, and Drew, the contestant, the audience, and all five rats could explode in a giant fireball. But we're building *mathematical models* here, and so we can afford to ignore remote possibilities like this.
- In addition, you can't describe the event 'the contestant wins' as a subset of the sample space. What if the pink rat finishes fifth? The contestant also has the orange rat. It is ambigious whether this possibility should be part of the event or not.

Advice: **Note that it is a very good thing to come up with wrong ideas** – provided that one then examines them critically, realizes that they won't work, and rejects them. Indeed, very often when solving a problem, your first idea will often be incorrect. Welcome this process – it is where the best learning happens.

This also means that you are not truly finished with a problem when you write down an answer. You are only finished when you think about your answer, check your work (if applicable), and make sure that your answer makes sense.

Problem 2. *Compute the probability that she wins both the car and the meal delivery.*

Here we care about the first *two* rats. We write

$$S = \{PO, PB, PR, PG, OP, OB, OR, OG, BP, BO, BR, BG, RP, RO, RB, RG, GP, GO, GB, GR\}.$$

The sample space has twenty elements in it. ($20 = 5 \times 4$: there are 5 possibilities for the first place finisher, and (once we know who wins) 4 for the second. More on this later.) The event that she wins is

$$\{PO, OP\}$$

and $P(E) = \frac{N(E)}{N(S)} = \frac{2}{20} = \frac{1}{10}$.

Problem 3. *Compute the probability that she wins all three prizes.*

Zero. Duh. She only won two rats! Sorry.

2.2 The Addition and Multiplication Rules

The Addition Rule (1). Suppose E and F are two *disjoint* events in the *same sample space* – i.e., they don't overlap. Then

$$P(E \text{ or } F) = P(E) + P(F).$$

Example 2.1 *You roll a die. Compute the probability that you roll either a 1, or a four or higher.*

Let $E = \{1\}$ *be the event that you roll a* 1*, and* $E = \{4, 5, 6\}$ *be the event that you roll a* 4 *or higher. Then*

$$P(E \text{ or } F) = P(E) + P(F) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 2.2 *You draw a poker card at random. What is the probability you draw either a heart, or a black card which is a ten or higher?*

Let E *be the event that you draw a heart. As before,* $P(E) = \frac{13}{52}$.
Let F *be the event that you draw a black card ten or higher, i.e.,*

$$F = \{A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit\}.$$

Then $P(F) = \frac{10}{52}$.
So we have

$$P(E \text{ or } F) = \frac{13}{52} + \frac{10}{52} = \frac{23}{52}.$$

Example 2.3 *You draw a poker card at random. What is the probability you draw either a heart, or a red card which is a ten or higher?*

This doesn't have the same answer, because hearts are red. If we want to apply the addition rule, we have to do so carefully.
Let E *be again the event that you draw a heart, with* $P(E) = \frac{13}{52}$.
Now let F *be the event that you draw a diamond which is ten or higher:*

$$F = \{A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit\}.$$

Now together E *and* F *cover all the hearts and all the red cards at least ten, and there is no overlap. So we can use the addition rule.*

$$P(E \text{ or } F) = P(E) + P(F) = \frac{13}{52} + \frac{5}{52} = \frac{18}{52}.$$

We can also use the addition rule with more than two events, as long as they don't overlap.

Example 2.4 *Consider the Rat Race contestant from earlier. What is the probability that she wins any two of the prizes?*

Solution 1. *We will give a solution using the addition rule. (Later, we will give another solution using the Multiplication Rule.)*

Recall that her chances of winning the car and the meal delivery were $\frac{1}{10}$*. Let us call this event* CM *instead of* E*.*

Now what are her chances of winning the car and the guitar? (Call this event CG*.) Again* $\frac{1}{10}$*. If you like, you can work this question out in the same way. But it is best to observe that there is a natural symmetry in the problem. The rats are all alike and any ordering is equally likely. They don't know which prizes are in which lanes. So the probability has to be the same.*

Finally, what is $P(MG)$*, the probability that she wins the meal service and the guitar? Again* $\frac{1}{10}$ *for the same reason.*

Finally, observe these events are all disjoint, because she can't possibly win more than two. So the probability is three times $\frac{1}{10}$*, or* $\frac{3}{10}$*.*

Here is a contrasting situation. Suppose the contestant had picked all three small prices correctly, and got to choose three of the rats. In this case, the probability she wins both the car and the meal service is $\frac{3}{10}$, rather than $\frac{1}{10}$. (You can either work out the details yourself, or else take my word for it.)

But this time the probability that she wins two prizes is *not* $\frac{3}{10} + \frac{3}{10} + \frac{3}{10}$, because now the events CM, CG, and MG are not disjoint: it is possible for her to win all three prizes, and if she does, then all of CM, CG, and MG occur!

It turns out that in this case the probability that she wins *at least* two is $\frac{7}{10}$, and the probability that she wins *exactly* two is $\frac{3}{5}$.

The Multiplication Rule. The multiplication rule computes the probability that two events E and F **both** occur. Here they are events in **different** sample spaces.

The formula is the following:

$$P(E \text{ and } F) = P(E) \times P(F).$$

It is not always valid, but it is valid in either of the following circumstances:

- The events E and F are *independent.*
- The probability given for F assumes that the event E occurs (or vice versa).

Example 2.5 *You flip a coin twice. What is the probability that you flip heads both times?*

We can use the multiplication rule for this. The probability that you flip heads if you flip a coin once is $\frac{1}{2}$*. Since coin flips are independent (flipping heads the first time doesn't make it more or less likely that you will flip heads the second time) we multiply the probabilities to get* $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Alternatively, we can give a direct solution. Let

$$S = \{HH, HT, TH, TT\}$$

and

$$E = \{HH\}.$$

Since all outcomes are equally likely,

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{4}.$$

We can also use the multiplication rule for more than two events.

Example 2.6 *You flip a coin twenty times. What is the probability that you flip heads every time?*

If we use the multiplication rule, we see at once that the probability is

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{20}} = \frac{1}{1048576}.$$

This example will illustrate the second use of the Multiplication Rule.

Example 2.7 *Consider the Rat Race example again (as it happened in the video). What is the probability that the contestant wins both the car and the meal service?*

Solution. The probability that she wins the car is $\frac{2}{5}$, as it was before. So we need to now compute the probability that she wins the meal service, *given that she won the car.*

This time the sample space consists of *four* rats: we leave out whichever one won the car. The event is that her remaining one rat wins the meal service, and so the probability of this event is $\frac{1}{4}$.

By the multiplication rule, the total probability is

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$

Example 2.8 *Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins all three prizes?*

Solution. The probability she wins the car is $\frac{3}{5}$, as before: the sample space consists of the five rats, and the event that she wins consists of the three rats she chooses. (Her probability is $\frac{3}{5}$ no matter which rats she chooses, under our assumption that they finish in a random order.)

Now assume that she wins the first prize. Assuming this, the probability that she wins the meals is $\frac{2}{4} = \frac{1}{2}$ The sample space consists of the four rats *other than the first place finisher,* and the event that she wins the meals consists of the two rats *other than the first place finishers.*

Now assume that she wins the first and second prizes. The probability she wins the guitar is $\frac{1}{3}$: the sample space consists of the three rats *other than the first two finishers,* and the event that she wins the meals consists of the single rat *other than the first two finishers.*

There is some subtlety going on here! To illustrate this, consider the following:

Example 2.9 *Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins the meal service?*

Solution. There are five rats in the sample space, she chooses three of them, and each of them is equally likely to finish second. So her probability is $\frac{3}{5}$ (same as her probability of winning the car).

But didn't we just compute that her odds of winning the car are $\frac{1}{2}$? What we're seeing is something we'll investigate much more later. This probability $\frac{1}{2}$ is a **conditional** probability: it assumes that one of the rats finished first, and illustrates what is hopefully intuitive: if she wins first place with one of her three rats, she is less likely to also win second place.

In particular, this reasoning illustrates the following misapplication of the multiplication rule. Suppose we compute again the probability that she wins all three prizes with three rats. She has a $\frac{3}{5}$ probability of winning first, a $\frac{3}{5}$ probability of winning second, and a $\frac{3}{5}$ probability of winning third. By the multiplication rule, the probability that all of these events occur is

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}.$$

What is wrong with this reasoning is that these events are *not independent.*

Michael Larson. Here is a bit of game show history. The following clip comes from the game show Press Your Luck on May 19, 1984.

`https://www.youtube.com/watch?v=UzggoA4lLwk`

Here Michael Larsen smashed the all-time record by winning $110,237. The truly fascinating clip starts at around 17:00, where Larson continues to press his luck, to the host's increasing disbelief. On 28 consecutive spins, Larson avoided all the whammies and each time hit a space that afforded him an extra spin. There are eighteen squares on the board, and on average there are approximately five spaces worth money and an extra spin.

Example 2.10 *Assume for simplicity that each time there are exactly five spaces (out of eighteen) that Larson wants to hit, and that the outcome is random and that each square is equally likely to occur.*

If Larson spins twenty-eight times, compute the probability that he hits a good spot every time.

Solution. *This is a straightforward application of the multiplcation rule. The answer is* $\left(\frac{5}{18}\right)^{28}$*, or approximately one in*

$$3,771,117,128,139,603.$$

Either Larson got very, *very*, **very**, **VERY** lucky....... or else the pattern is not random and he figured it out.

Card Sharks. Here is another game show from the eighties that leads to interesting probability computations.

Game Description (Card Sharks): Each of two contestants receives a lineup of five cards. The first is shown to each contestant, and a *marker* is placed on the first card. The objective of each **round** is to reach the last card.

A **turn** by the contestant consists of the following. She starts with the (face-up) card at the marker, and may replace it with a random card if she chooses. She then guesses whether the next card is higher or lower, which is then revealed.

If is the last card and her guess is correct, she wins the round. Otherwise, she may keep guessing cards for as long as she likes untill one of three things happens: (1) she guesses the last card correctly, and wins; (2) she guesses any card incorrectly, in which case the cards she has guessed are all discarded and replaced with new cards (face down); (3) she chooses to end the turn by moving her marker forward to the last card guessed correctly.

The **round** begins with a trivia question (I don't describe the rules for that here), and the winner gets to take a turn. If this turn ends with a freeze, the contestants go to another trivia question; if it ends with a loss, the other contestant takes a turn.

There is also a *bonus round* which we won't discuss here. (We could though; analyzing this would make an interesting term project.)

Here[1] is a typical clip:

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[1] *Summary of the clip*: (**Please note.** The trivia questions are off-color and arguably sexist. This is unfortunately common on this show.) The contestants are Royce and Cynthia. Cynthia wins the first trivia question. Her initial card is a king. She keeps it and guesses lower; the second card is a two. She guesses higher; the third card is a nine. She freezes on position three.

Royce wins the next trivia question. His initial card is an eight; he changes it and gets a four. He guesses higher; the second card is a six. He guesses higher; the third card is a nine. He freezes on position three.

Royce wins the next trivia question. He starts on position three and chooses to replace the nine, and gets a three. He guesses higher; the fourth card is a five. He guesses higher; the fifth card is a king and Royce wins the round.

`https://www.youtube.com/watch?v=bUv0CRU6t5o`

Here is our objective: *Assuming that the trivia questions are a 50-50 tossup, determine the optimal strategy in all situations.* This problem is somewhat difficult (and our mental heuristics for it are fairly spot on). But at least in principle, it is possible to give a complete solution to this problem.

We won't try to achieve this all at once. Instead, we'll ask a number of probability questions to get started:

Example 2.11 *Consider Cynthia's first turn, where she guesses 'lower'. Compute the probability that she is correct.*

Answer. *The sample space consists of the* 51 *cards other than the king of clubs. Of these, only seven are not lower: the four aces, and the three remaining kings. So* $51 - 7 = 44$ *cards are lower, and her chances are* $\frac{44}{51}$.

We also compute the probabilities at the next two rounds. She guesses the third card will be higher than a 2. There are 50 cards remaining, and 47 of them are higher than a 2, so her odds are $\frac{47}{50}$.

The next card was a 9. Of the 49 remaining cards, 27 are lower than a 9 and 19 are higher. (And the three remaining nines are neither higher nor lower – so she would lose no matter what she picked). If she chose to play, her odds of winning the next card would be $\frac{27}{49}$, or slightly better than 50-50. She quite reasonably chooses to freeze and lock in her position.

Now we skip ahead to Royce's second round (when both Royce and Cynthia have frozen on the third of five cards).

Here are several questions we can ask:

- Given that Royce has replaced his nine with a three, compute the probability that he can win the round (assuming he doesn't freeze).
- Before Royce sees the three, compute the probability that he can win the round.
- Given that Royce's card is a five, compute the probability that he wins if he doesn't choose to freeze.
- If Royce chooses to freeze, answers the next trivia question correctly, and gets to go again, compute the probability that he wins on his next attempt.

  (Note that this is not the total probability he wins: he could lose on his next attempt, but then answer another trivia question correctly and get yet another try.)
- If Royce chooses to freeze and Cynthia answers the next trivia question correctly, what is the probability that she wins the next round (if she doesn't freeze)?

These questions get us closer to the question we're *really* interested in: should Royce freeze on the five or not? As is often the case, the question we are interested in is quite difficult and we build up to being able to answer it.

We tackle the first question.

Example 2.12 *Given that Royce has replaced his nine with a three, compute the probability that he can win the round. Assume that he doesn't choose to freeze, and that his higher/lower guess is always optimal.*

Note that there are 48 cards left in the deck: a three, a four, a six, and a nine are all missing.

It is easy to compute the probability that Royce's *first* guess is correct: out of 48 remaining cards, 41 are higher, so the probability is $\frac{41}{48}$. Now, **assuming that Royce's first guess is correct**, what is the probability that his second guess is correct?

Well ...... we don't know. It depends on what the first card **is**. Later, we will see some clever tricks for carrying out this sort of computation more easily. But for now, we outline a 'brute force' computation:

- Royce's first guess will be correct if the first card is a four, five, six, seven, eight, nine, ten, jack, queen, king, or ace.
- Based on Royce's first guess, we can determine what Royce should guess for the second card and the probability that this guess will be correct.

  Let's do an example of this. Suppose the first card is a four; the probability of this occurring is $\frac{3}{48}$. (This reduces to $\frac{1}{16}$, but the pattern will be clearer if we do not reduce our fractions to lowest terms.)

  Then Royce should clearly guess that the second will be higher. There are 47 remaining cards, of which 38 are higher than a four. So *assuming that the first card is a four*, the probability that Royce wins is $\frac{38}{47}$. Therefore, the probability that *the first card is a four* **and** *Royce wins* is $\frac{3}{48} \times \frac{38}{47}$.
- We will therefore use **both** the addition and the multiplication rules by **dividing into cases**: For each possible first card n (that doesn't lose Royce the round immediately), we compute the probability that the first card **is** n **and that** Royce wins the round. This is the multiplication rule.

  Since all of these possibilities are mutually exclusive, but one of them has to occur if Royce is to win, we see that the probability that Royce wins is the total of the probabilities we computed in the first step. This is the addition rule!

Let's roll up our sleeves and do it. The proof won't be pretty, but it is not as scary as it looks.

- With probability $\frac{3}{48}$ the first card will be a four. Then Royce should guess higher, and with probability $\frac{38}{47}$ the next card will be higher.

- With probability $\frac{4}{48}$ the first card will be a five. Then Royce should guess higher, and with probability $\frac{34}{47}$ the next card will be higher.
- With probability $\frac{3}{48}$ the first card will be a six. Then Royce should guess higher, and with probability $\frac{31}{47}$ the next card will be higher.
- With probability $\frac{4}{48}$ the first card will be a seven. Then Royce should guess higher, and with probability $\frac{27}{47}$ the next card will be higher.
- With probability $\frac{4}{48}$ the first card will be an eight. Then Royce should guess higher, and with probability $\frac{23}{47}$ the next card will be higher.
- With probability $\frac{3}{48}$ the first card will be a nine. Then Royce should guess lower, and with probability $\frac{24}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a ten. Then Royce should guess lower, and with probability $\frac{27}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a jack. Then Royce should guess lower, and with probability $\frac{31}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a queen. Then Royce should guess lower, and with probability $\frac{35}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a king. Then Royce should guess lower, and with probability $\frac{39}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be an ace. Then Royce should guess lower, and with probability $\frac{43}{47}$ the next card will be lower.

(Note that all of the cases look more or less the same. Often, this is an indication that you can look for shortcuts – but we won't do so here.)

The total probability that Royce wins is therefore

$$\frac{3}{48}\cdot\frac{38}{47}+\frac{4}{48}\cdot\frac{34}{47}+\frac{3}{48}\cdot\frac{31}{47}+\frac{4}{48}\cdot\frac{27}{47}+\frac{4}{48}\cdot\frac{23}{47}+\frac{3}{48}\cdot\frac{24}{47}+\frac{4}{48}\cdot\frac{27}{47}+\frac{4}{48}\cdot\frac{31}{47}+\frac{4}{48}\cdot\frac{35}{47}+\frac{4}{48}\cdot\frac{39}{47}+\frac{4}{48}\cdot\frac{43}{47}.$$

This is equal to $\frac{1315}{2256}$, which is already in lowest terms. Yeah, I know. You were hoping it would be nice and simple, and that in retrospect you could have solved the problem in your head. You couldn't have. Neither could I. Sometimes math is like that.

This is roughly 58.2%, which is not bad at all.

2.3 Permutations and factorials

This video[2] illustrates a playing of the Price Is Right game **Ten Chances**:

`https://www.youtube.com/watch?v=iY_gmGcDKXE`

Game Description (Ten Chances (The Price Is Right)): The contestant is shown a small prize, a medium prize, and a large prize. She has ten chances to win as many prizes as she can.

The price of small prize has two numbers in it, and the contestant is shown three different numbers. She then guesses the price of the first prize. She takes as many chances as she needs to.

Once she wins the small prize, she attempts to win the medium prize. The price of the medium prize has three numbers in it, and the contestant is shown four.

Finally, if she wins the medium prize, she attempts to win the car. Its price has five numbers in it, and the contestant is shown these five.

Example 2.13 *The price of the pasta maker contains two digits from* $\{0, 6, 9\}$*. Suppose that each possibility is equally likely to be the price of the pasta maker.*

If the contestant has one chance, what are her odds of winning?

Solution 1. We can give a straightforward solution by simply enumerating the sample space of all possibilities. It is

$$\{06, 09, 60, 69, 90, 96\}.$$

The contestant's choice describes an event with one of these possibilities in it. Since we hypothesized that each was equally likely to occur, her odds of winning are $\frac{1}{6}$.

Solution 2. We use the multiplication rule. There are three different possibilities for the first digit, and exactly one of them is correct. The probability that she gets the first digit correct is therefore $\frac{1}{3}$.

Now, **assume she got the first digit correct.** (If she didn't, she might have used up the correct second digit already, and be doomed to botch that one also!) Then there are two remaining digits, and the probability that she picks the correct one is $\frac{1}{2}$.

Thus the probability of getting both correct is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Notice, incidentally, that our assumption that the possibilities are equally likely is not realistic. Surely the pasta maker's price is not 06 dollars? Especially since you'd write it 6

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[2] *Summary of the clip*: She plays Ten Chances for a pasta maker, a lawnmower, and a car. The digits in the pasta maker are 069, and she guesses the correct price of 90 on her second chance. The digits in the mower are 0689, and she guesses the correct price of 980 on her third chance. (Her third chance overall; she took only once to win the mower.) The digits in the car are 01568, and she guesses the correct price of 16,580 on her first try (and wins).

Barker then hides beyond the prop ... and, uh, (**please note**) the contestant violates his personal space.

and not 06? (Indeed, if you have watched the show a lot, you know that when there is a zero the price always ends with it. Knowing this fact is a *big* advantage.)

Now, she is going to use up at most six of her chances on the pasta maker, so she gets to move on to the mower. Here the price contains three digits from $\{0, 6, 8, 9\}$. This problem can be solved in the same way. The relevant sample space is

$$\{068, 069, 086, 089, 096, 098, 608, 609, 680, 689, 690, 698, 806, 809, 860, 869, 890, 896, 906, 908, 960, 968, 980, 986$$

which has 24 elements in it, so her probability of winning is $\frac{1}{24}$. The analogue of solution 2 gives $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$.

Finally, the price of the car has the digits $\{0, 1, 5, 6, 8\}$ and this time she uses all of them. The sample space is too long to effectively write out. So we work out the analogue of Solution 2: Her odds of guessing the first digit are $\frac{1}{5}$. If she does so, her odds of guessing the second digit is $\frac{1}{4}$ (since she has used one up). If both these digits are correct, her odds of guessing the third digit is $\frac{1}{3}$. If these three are correct, her odds of guessing the fourth digit are $\frac{1}{2}$. Finally, **if** the first four guesses are correct then the last digit is automatically correct by process of elimination. So the probability she wins is

$$\frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{120}.$$

Here the number 120 is equal to 5!, or 5 **factorial**. In math, an exclamation point is read 'factorial' and it means the product of all the numbers up to that point. We have

$$\begin{aligned}
1! &= 1 &&= 1\\
2! &= 1 \times 2 &&= 2\\
3! &= 1 \times 2 \times 3 &&= 6\\
4! &= 1 \times 2 \times 3 \times 4 &&= 24\\
5! &= 1 \times 2 \times 3 \times 4 \times 5 &&= 120\\
6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 &&= 720\\
7! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 &&= 5040\\
8! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 &&= 40320\\
9! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 &&= 362880\\
10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 &&= 3628800,
\end{aligned}$$

and so on. We also write $0! = 1$. Why 1 and not zero? 0! means 'don't multiply anything', and we think of 1 as the starting point for multiplication. (It is the *multiplicative identity*, satisfying $1 \times x = x$ for all x.) So when we compute 0! it means we didn't leave the starting point.

These numbers occur **very** commonly in the sorts of questions we have been considering, for reasons we will shortly see.

Example 2.14 *The lucky contestant wins the first two prizes in only three chances, and has seven chances left over. If each possibility for the price of the car is equally likely, then what is the probability that she wins it?*

The answer is seven divided by $N(S)$, the number of elements in the sample space. So if we could just compute $N(S)$, we'd be done.

Here there is a trick! She guesses 16580, and we know that the probability that this is correct is $\frac{1}{N(S)}$: one divided by the number of total possible guesses. But we already computed the probability: it's $\frac{1}{120}$. Therefore, we know that $N(S)$ is 120, without actually writing it all out!

The mathematical discipline of **combinatorics** is the art of *counting without counting.* We just solved our first combinatorics problem: we figured out that there were 120 ways to rearrange the numbers $0, 1, 5, 6, 8$ without actually listing them. We now formalize this principle.

Definition 2.15 *Let T be a* **string***. For example,* 01568 *and* 22045 *are strings of numbers, ABC and xyz are strings of letters, and* $\otimes - \oplus$♣☠ *is a string of symbols. Order matters:* 01568 *is not the same string as* 05186*.*

A **permutation** *of T is any reordering of T.*

So, for example, if T is the string 1224, then 2124, 4122, 1224, and 2142 are all permutations of T. Note we *do* consider T itself to be a permutation of T, for the same reason that we consider 0 a number. It is called the **trivial permutation**.

We have the following:

Proposition 2.16 *Let T be a string with n distinct symbols. Then there are exactly $n!$ distinct permutations of T.*

In math, a *proposition* (or a *theorem*) is a statement of something true. We have stated lots of true facts in these notes; here the title 'Proposition' indicates that this one is particularly important and worth your attention.

Please read the statement carefully. In particular, the conclusion is only guaranteed to hold when the hypotheses also hold. If the hypotheses don't hold, then the conclusion may or may not be true. For example, if T is the string 122, then the set of all permutations of it is

$$\{122, 212, 221\}$$

which has 3 elements, and $3 \neq 3! = 6$.

Note also that this solves our earlier Ten Chances question. The contestant's guesses are all permutations of the string 01568, of which there are $5! = 120$. The sample space S consists of all 120 permutations. The contestant can make seven guesses, so let E be the set of these 7 permutations. Since we have assumed that each possible guess is equally likely to be correct, her odds (probability) of winning are $\frac{7}{120}$.

We will now offer a **proof** of the proposition. Please don't be too scared by the word 'proof': it just means a convincing explanation of why it is true. This course will not focus on *writing* proofs, but it is good to gain practice reading them.

Proof: Suppose T is a string with n distinct symbols, and we want to construct a permutation of T. symbol We first choose the first symbol. Since T has n distinct symbols, we have n choices for the first symbol.

No matter what we choose for the first symbol, there are $n - 1$ choices for the second symbol (all but the one we picked already), so that there are $n \times (n - 1)$ choices for the first two.

Similarly, there are $n - 2$ choices for the third symbol, and so on. This continues until the last (the nth) symbol, for which there is exactly one choice. $\Box$

In math we often end proofs with a little square. If you like, you can end proofs with the phrase **QED**, which is an abbreviation for 'quod erat demonstrandum' – Latin for 'that which was to be shown'. In practice, saying or writing 'QED' serves the same purpose as a football player spiking the ball after he has scored a touchdown.

If you are especially observant, you will notice that the proof is very similar to our explanation of the multiplication rule for probability. There is a good reason for this: the same principle underlies both, and counting and probability are two sides of the same coin.

We now return to our Ten Chances contestant. Recall that she has seven chances to win the car.

Example 2.17 *Suppose that the contestant has watched The Price Is Right a lot and so knows that the last digit is the zero. Compute the probability that she wins the car, given seven chances.*

Solution. Here her possible guesses consist of permutations of the string 1568, followed by a zero. There are $4! = 24$ of them, so her winning probability is $\frac{7}{24}$.

Her winning probability went up by a factor of exactly 5 – corresponding to the fact that $\frac{1}{5}$ of the permutations of 01568 have the zero in the last digit. Equivalently, a random permutation of 01568 has probability $\frac{1}{5}$ of having the zero ias the last digit.

Now, a smart contestant can do better. Suppose, for example, that she guessed 85610. Mathematically it looks like a good guess .... but she is playing for a Chevy Cavalier. I mean, really. We can rule out the 8 as the first digit, as well as the 6 and the 5.

Example 2.18 *Suppose that the contestant knows that the the last digit is the zero and the first digit is the one. Compute the probability that she wins the car, given seven chances.*

Solution. Her guesses now consist of permutations of the string 568, with a 1 in front and followed by a zero. There are $3! = 6$ of them. Assuming that the assumptions are correct and that she doesn't screw up, she is a sure bet to win the car.

Mathematically, her probability of winning is 1 (which is the same as 100%). Please *don't* answer that her probability is $\frac{7}{6}$. This doesn't make much sense!

Note that it is only true of Ten Chances that car prices always end in zero – not of The Price Is Right in general. Here is a contestant who is very excited until she realizes the odds she is against:

`https://www.youtube.com/watch?v=AAIU6knD7BA`

2.4 Exercises

Most of these should be relatively straightforward, but there are a couple of quite difficult exercises mixed in here for good measure.

1. Card questions. In each question, you choose at random a card from an ordinary deck. What is the probability you –
    (a) Draw a spade?
    (b) Draw an ace?
    (c) Draw a face card? (a jack, queen, king, or an ace)
    (d) Draw a spade or a card below five?
2. Dice questions:
    (a) You roll two dice and sum the total. What is the probability you roll exactly a five? At least a ten?

    Solution. The sample space consists of 36 possibilities, 11 through 66. The first event can be described as $\{14, 23, 32, 41\}$ and has probability $\frac{4}{36} = \frac{1}{9}$. The second can be described as $\{46, 55, 64, 56, 65, 66\}$ and has probability $\frac{6}{36} = \frac{1}{6}$.

    (b) You roll three dice and sum the total. What is the probability you roll at least a 14? (This question is kind of annoying if you do it by brute force. Can you be systematic?)

    Solution. There are several useful shortcuts. Here is a different way than presented in lecture. The sample space consists of $6 \times 6 \times 6 = 216$ elements, 111 through 666. The event of rolling at least a 14 can be described as

    $$\{266(3), 356(6), 366(3), 446(3), 455(3), 456(6), 466(3), 555(1), 556(3), 566(3), 666(1)\}.$$

    The number in parentheses counts the number of permutations of that dice roll, all of which count. For example, 266, 626, and 662 are the permutations of 266. There are 35 possibilities total, so the probability is $\frac{35}{216}$.

(c) The dice game of *craps* is (in its most basic form) played as follows.

You roll two dice. If you roll a 7 or 11 on your first roll, you win immediately, and if you roll a 2, 3, or 12 immediately, you lose immediately. Otherwise, your total is called "the point" and you continue to roll again until you roll either the point (again) or a seven. If you roll the point, you win; if you roll a seven, you lose.

In a game of craps, compute the probability that you win on your first roll and the probability that you lose on your second roll.

Solution. The probability of winning on your first roll is the probability of rolling a 7 or 11: $\frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$.

For the second question, I intended to ask the probability that you lose on your *first* roll. Oops. Let's answer the question as asked. There are multiple possible interpretations, and here is one. Let us compute the probability that you lose on the second round, presuming that the game goes on to a second round. This is the probability of rolling a 6 or $\frac{1}{6}$.

(d) In a game of craps, compute the probability that the game goes to a second round and you win on the second round.

Solution. This can happen in one of six possible ways: you roll a 4 twice in a row, a 5 twice in a row, or similarly with a 6, 8, 9, or 10.

The probability of rolling a 4 is $\frac{3}{36}$, so the probability of rolling a 4 twice in a row is $\left(\frac{3}{36}\right)^2$. Similarly with the other dice rolls; the total probability is

$$(1)\quad \left(\frac{3}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{3}{36}\right)^2 = \frac{9+16+25+25+16+9}{1296} = \frac{100}{1296} = \frac{25}{324}.$$

(e) In a game of craps, compute the probability that the game goes to a second round and you lose on the second round.

Solution. Multiply the probability that the game goes onto a second round (easily checked to be $\frac{2}{3}$) by the probability $\frac{1}{6}$ computed earlier, so $\frac{1}{9}$.

(f) In a game of craps, compute the probability that you win.

Solution. With probability $\frac{2}{9}$ you win on your first round. We will now compute the probability that you win later, with the point equal to n, for n equal to 4, 5, 6, 8, 9, or 10. We will then add these six results. Write the probability of rolling n on one roll of two dice as $\frac{a}{36}$, so that a is 3, 4, or 5 depending on n.

- As we computed before, the probability of winning on the second round (with point n) is $\left(\frac{a}{36}\right)^2$.

- On each round after the first, there is a probability $\frac{30-a}{36}$ of rolling something other than 7 or the point. This is the probability that the game goes on to another round.
- So, the probability of winning on the third round is the probability of: rolling the point on the first round, going another turn in the second round, rolling the point on the third round. This is $\left(\frac{a}{36}\right)^2 \cdot \left(\frac{30-a}{36}\right)$.
- Similarly, the probability of winning with point n on the fourth round is $\left(\frac{a}{36}\right)^2 \cdot \left(\frac{30-a}{36}\right)^2$, and so on. The total of all these probabilities is

$$\left(\frac{a}{36}\right)^2 \sum_{k=0}^{\infty} \left(\frac{30-a}{36}\right)^k.$$

- For $|r| < 1$, we have the infinite sum formula $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Plugging this in, the above expression is

$$\left(\frac{a}{36}\right)^2 \cdot \frac{36}{6+a} = \frac{a^2}{36(6+a)}.$$

So we add this up for $a = 3$ (twice, for $n = 4$ or 5), $a = 4$ (twice), and $a = 5$ (twice). We get

$$2 \cdot \left(\frac{9}{36 \cdot 9} + \frac{16}{36 \cdot 10} + \frac{25}{36 \cdot 11}\right) = \frac{134}{495}.$$

Adding the to the first round probability of $\frac{2}{9}$ we get

$$\frac{2}{9} + \frac{134}{495} = \frac{244}{495}.$$

This is a little less than a half. As expected, the house wins.

3. Consider the game Press Your Luck described above. Assume (despite rather convincing evidence to the contrary) that the show is random, and that you are equally likely to stop on any square on the board.

   (a) On each spin, estimate the probability that you hit a Whammy. Justify your answer.

   (Note: This is mostly not a math question. You have to watch the video clip for awhile to answer it.)

   (b) On each spin, estimate the probability that you *do not* hit a Whammy.

   (c) If you spin three times in a row, what is the probability you don't hit a whammy? Five? Ten? Twenty-eight? (If your answer is a power of a fraction, please also use a calculator or a computer to give a decimal approximation.)

4. Consider the game Rat Race described above.

(a) Suppose that the contestant only prices one item correctly, and so gets to pick one rat. What is the probability that she wins the car? That she wins *something*? That she wins nothing?

(b) What if the customer prices all three items correctly? What is the probability that she wins the car? Something? Nothing? All three items?

(c) Consider now the first part of the game, where the contestant is pricing each item. *Assume* that she has a 50-50 chance of pricing each item correctly. What is the probability she prices no items correctly? Exactly one? Exactly two? All three? Comment on whether you think this assumption is realistic.

Solution. Foobar.

(d) Suppose now that she has a 50-50 chance of pricing each item correctly, and she plays the game to the end. What is the probability she wins the car?

3 Expectation

3.1 Definitions and examples

We come now to the concept of **expected value**. We will give a few simple examples and then give a formal definition.

Example 3.1 *You play a simple dice game. You roll one die; if it comes up a six, you win 10 dollars; otherwise you win nothing. On average, how much do you expect to win?*

Solution. Ten dollars times the probability of winning, i.e.,

$$10 \times \frac{1}{6} = 1.66\ldots$$

So, for example, if you play this game a hundred times, on average you can expect to win 100 dollars.

Example 3.2 *You play a variant of the dice game above. You roll one die; if it comes up a six, you still win 10 dollars. But this time, if it doesn't come up a six, you lose two dollars. On average, how much do you expect to win?*

Solution. We take into account both possibilities. We multiply the events that you win 10 dollars or lose 2 dollars and multiply them by their probabilities. The answer is

$$10 \times \frac{1}{6} + (-2) \times \frac{5}{6} = 0.$$

On average you expect to break even.

Definition 3.3 *Consider a random process whose outcome can be described as a real number. Suppose that the possible outcomes are* $a_1, a_2, \ldots a_n$*, which occur with respective probabilities* $p_1, p_2, \ldots, p_n$*. Then the* **expected value** *of this process is*

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k.$$

If the outcomes represent the amount of money you win (positive) or lose (negative), then the expected value is the amount you should expect to win on average.

Example 3.4 *You roll a die and win a dollar amount equal to your die roll. Compute the expected value of this game.*

Solution. The possible outcomes are that you win 1, 2, 3, 4, 5, or 6 dollars, and each happens with probability $\frac{1}{6}$. Therefore the expected value is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{21}{6} = 3.5.$$

As another example, consider the Deal or No Deal clip from the introduction.

`https://www.youtube.com/watch?v=I3BzYiCSTo8`

This is quite simple to analyze, and indeed we did so in the introduction.

For example, after the second round, he has eliminated 11 briefcases and 15 remain, which contain a total of \$2,808,416, or an average of \$187,227. If he keeps playing all the way until the end, the expected value is equal to the average of the remaining briefcases. The bank offers him a flat payment of \$125,000 to quit. If he wants to maximize his expected value, he should refuse this, and indeed he does.

We now consider some expected value computations arising from the popular game show **Wheel of Fortune**.

Game Description (Wheel of Fortune, Simplified Version): The contestants play several rounds where they try to solve word puzzles and win money. (The contestant who has won the most money then gets to play in a bonus round.)

The puzzle consists of a phrase whose letters are all hidden. In turn, each contestant either **attempts to solve the puzzle** or **spins the wheel**. If the contestant attempts to solve, he states a guess; if is correct, he wins all the money in his bank, and if it is wrong, play passes to the next player.

The wheel contains lots of spaces with various dollar amounts or the word 'bankrupt'. When the contest spins, the wheel comes to rest on one of these spaces. If 'bankrupt', the contestant loses all his money from this round and play passes to the next contestant.

Otherwise, the contestant chooses a letter. If that letter appears in the puzzle (and has not yet been guessed), then each of these letters is revealed and the contestant wins the amount of money on his space for each time it appears. If the letter does not appear, the contestant wins nothing and play passes to the next contestant.

These rules are incomplete: the contestants can 'buy a vowel'; there are non-monetary prizes on the board which work differently (you don't win more than one of them if a letter appears multiple times), other spaces like 'lose a turn', and so forth.

Consider the episode of Wheel of Fortune shown in this clip:

`https://www.youtube.com/watch?v=A8bZUXi7zDE`

Robert wins the first round in short order. After guessing only two letters (and buying a vowel) he chooses to solve the puzzle. Was his decision wise?

Let us make some assumptions to simplify the problem and set up an expected value computation:

- Robert wants to maximize the expected value of his winnings this round.

  This is not completely accurate, especially in the final round; the contestants are interested in winning *more than the other two contestants*, because the biggest winner gets to play the bonus round. But it is reasonably close to accurate, especially early in the running.

- Robert definitely knows the solution to the puzzle.

  So, if he chooses to spin again, it's to rack up the amount of prizes and money he wins.

- If Robert loses his turn, then he won't get another chance and will therefore lose everything.

  In fact, there is a chance that each of the other two contestants will guess wrongly or hit the 'bankrupt' or 'lose a turn' spots on the wheel. But this puzzle doesn't look hard: the first word *don't* is fairly obvious; also, the second word looks like *bet*, *get*, or *let* and B, G, and L are all in the puzzle. Robert is wise to assume he won't get another chance.

- We won't worry too much about the 'weird' spots on the board.

  The $\frac{1}{3}$-sized million dollar wedge is not what it looks like: it sits over (what I believe is) a \$500 wedge now, and offers the contestant the opportunity to win \$1,000,000 in the bonus round *if* he goes to the bonus round *and* doesn't hit bankrupt before then *and* solves the bonus puzzle correctly *and* chooses the million dollars randomly as one of five prizes. It's a long shot, although three contestants have indeed won the million.

So we freeze-frame the show and we count what we see. Out of 24 wedges on the wheel, there are:

- 16 ordinary money wedges on the wheel, with dollar amounts totalling $12,200.
- Two 'bankrupt' wedges, a 'lose a turn' wedge, and an additional two thirds of a bankrupt wedge surrounding the million.
- A one-third size wedge reading 'one million'.
- The cruise wedge. This isn't relevant to the contestant's decision, because he wins the cruise and reveals an ordinary wedge underneath. We can't see what it is, so let's say $500.
- Two other positive wedges.

Let us now compute the expected value of another spin at the wheel. There are (with the cruise wedge) 17 ordinary wedges worth a total of $12,700. If the contestant hits 'bankrupt' or 'lose a turn' he loses his winnings so far ($10,959 including the cruise). Let us guess that the million wedge is worth, on average, $5,000 to the contestant and that the other two are worth $2,000 each. His expected value from another spin is

$$\frac{1}{24} \cdot 12700 + \frac{2\frac{2}{3}}{24} \cdot (-10959) + \frac{2}{24} \cdot 2000 + \frac{\frac{1}{3}}{24} \cdot 5000 = -\$452.39.$$

It is clear by a large margin to solve the puzzle and lock in his winnings.

Remark 3.5 *You may be wondering where the* $\frac{1}{24} \cdot 12700$ *came from. Here is one way to see it: the seventeen wedges have an average of* $\frac{12700}{17}$ *dollars each, and there is a* $\frac{17}{24}$ *probability of hitting one of them. So the contribution is*

$$\frac{12700}{17} \times \frac{17}{24} = \frac{12700}{24}.$$

Now let us suppose that there was some consonant appearing in the puzzle twice. In that case Robert would know that he could guess it and get *double* the amount of money he spun. So, in our above computation, we double the 12700. (We should probably increae the 2000 and 5000 a little bit, but not double them. For simplicity's sake we'll leave them alone.) In this case the expected value of spinning again is

$$\frac{1}{24} \cdot 12700 \cdot 2 + \frac{2\frac{2}{3}}{24} \cdot (-10959) + \frac{2}{24} \cdot 2000 + \frac{\frac{1}{3}}{24} \cdot 5000 = -\$76.77,$$

so slightly positive. If Robert has the stomach to risk his winnings so far, he should consider spinning again.

For an example where Robert arguably chooses unwisely, skip ahead to 10:45 on the video (the third puzzle) where he solves the puzzle with only $1,050 in the bank. In the exercises, you are asked to compute the expected value of another spin. Note that there are now two L's and two R's, so he can earn double the dollar value of whatever he lands on.

There is now a \$10,000 square on the wheel, and hitting 'bankrupt' only risks his \$1,050. (His winnings from the first round are safe.)

There is one factor in favor of solving now: an extra prize (a trip to Bermuda) for the winner of the round. If it were me, I would definitely risk it. You do the math, and decide if you agree.

(But see the fourth run, where I would guess he knows the puzzle and is running up the score.)

The game **Punch a Bunch** from The Price Is Right has a similar (but much simpler) mechanic:

Game Description (Punch-a-Bunch (The Price Is Right)): The contestant is shown a punching board which contains 50 slots with the following dollar amounts: 100 (5), 250 (10), 500 (10), 1000 (10), 2500 (8), 5000 (4), 10,000 (2), 25,000 (1). The contestant can earn up to four punches by pricing small items correclty. For each punch, the contestant punches out one hole in the board.

The host proceeds through the holes punched one at a time. The host shows the contestant the amount of money he has won, and he has the option of either taking it and ending the game, or discarding and going on to the next hole.

So, if you just get one punch, there is no strategy: you just take whatever you get. In this case the expected value is the total of all the prizes divided by 50, or $\frac{103000}{50} = 2060$.

Here is a typical playing of Punch-a-Bunch:

`https://www.youtube.com/watch?v=25THBiZNPpo`

The contestant gets three punches, throws away 500 on his first punch, 1000 on his second, and gets 10,000 on his third. Was he right to throw away the 1000?

Clearly yes, as the expected value of one punch is 2,060. Indeed, in this example it is a little bit higher: there is \$101,500 in prizes left in 48 holes, for an average of \$2,114.58. You don't have to do the math exactly: just remember that two of the small prizes are gone, so the average of the remaining ones goes up slightly.

So let's figure out optimal strategy for this game. The last two rounds are easy.

- On your last round, there is no strategy: you take whatever you get.
- On your next-to-last round, throw away anything less than \$2500. You should keep the \$2500 prize if you're trying to maximize your expected value. It's pretty close though; I wouldn't fault anyone who tried for the big prize. (If nothing else, it would make better TV.)
- What about your third-to-last round?

We are going to compute the expected value of the *next-to-last round*. We'll assume that this is also the contestant's *first* round; otherwise, the contestant will have thrown away one or two small prizes and the expected value will be slightly higher. (This is another example of where we simplify our problem by making such an assumption. In this case, the assumption is very nearly accurate.)

- The contestant might win \$25,000 ($\frac{1}{50}$ chance), \$10,000 ($\frac{2}{50}$ chance), \$5,000 ($\frac{4}{50}$ chance), or \$2,500 ($\frac{8}{50}$ chance). As we discussed earlier, the contestant should keep it and end the game.
- The contestant might draw a card less than \$2,500 ($\frac{35}{50}$ chance). As we discussed earlier, the contestant should throw it away. In this case, the contestant expects to win \$2,060 (in fact, slightly more, as previously discussed) on average from the last punch.

So the expected value of the next-to-last round is

$$25000 \cdot \frac{1}{50} + 10000 \cdot \frac{2}{50} + 5000 \cdot \frac{4}{50} + 2500 \cdot \frac{8}{50} + 2060 \cdot \frac{35}{50} = \$3,142.$$

So we see that on the contestant's third-to-last round, he should throw away the \$2,500 cards in addition to everything cheaper, and only settle for \$5,000 or more. The expected value of the third-to-last round is

$$25000 \cdot \frac{1}{50} + 10000 \cdot \frac{2}{50} + 5000 \cdot \frac{4}{50} + 3142 \cdot \frac{43}{50} = \$4,002.12.$$

Therefore, if the contestant gets four punches, his strategy on the first round should be the same: to keep anything \$5,000 or more, and throw everything else away. The expected value of a four-round game is

$$25000 \cdot \frac{1}{50} + 10000 \cdot \frac{2}{50} + 5000 \cdot \frac{4}{50} + 4002 \cdot \frac{43}{50} = \$4,741.72.$$

A contestant can win only up to four punches. But we see that if the contestant got more, he would eventually throw away the \$5,000 cards too.

Who Wants To Be a Millionaire?

Here is a typical clip from Who Wants To Be a Millionaire:

`https://www.youtube.com/watch?v=sTGxOqp3qB8`

The rules in force for this episode were as follows.

Game Description (Who Wants to be a Millionaire?): The contestant is provided with a sequence of 15 trivia questions, each of which is multiple choice with four possible answers. They are worth an increasing amount of money: 100, 200, 300, 500, and then (in thousands) 1, 2, 4, 6, 16, 32, 64, 125, 250, 500, 1000. (In fact, in this epsiode, the million dollar question was worth \$2,060,000.)

At each stage he is asked a trivia question for the next higher dollar amount. He can choose to answer, or to not answer and to keep his winnings so far. If he answers correctly, he goes to the next level. If he answers incorrectly, the game is over. At the $1,000 and $32,000 level his winnings are protected: he is guaranteed of winning at least that much money. Beyond that, he forfeits any winnings if he ventures an incorrect answer.

He has three 'lifelines', each of which may be used exactly once over the course of the game: '50-50', which eliminates two of the possible answers; 'phone a friend', allowing him to call a friend for help; and 'ask the audience', allowing him to poll the audience for their opinion.

In general we want to ask the following question:

Question. The contestant is at level x, and (after using any applicable lifelines) estimates that he has a probability δ of answering correctly. Should he guess or not?

Let us assume that $x \geq 32000$ (that's the interesting part of the show). Note that if $x = 32000$, he should always guess since he is risking nothing.

Suppose then that $x = 64000$, and for now we'll consider *only the next question.* We will work with δ as a variable, and so our answer will be of the form 'He should guess if he believes his probability of answering correctly is greater than [something].' His winnings will be 32000 if he is incorrect and 125000 if he is right; and these events have probability $1 - \delta$ and δ respectively. Therefore, the expected value of guessing is

$$(1 - \delta) \cdot 32000 + \delta \cdot 125000 = 32000 + \delta \cdot 93000.$$

When is this greater than 64000? We solve the inequality $32000 + 93000\delta > 64000$, which is equivalent to $93000\delta > 32000$, or $\delta > \frac{32000}{93000} = \frac{32}{93}$. This is a little bit bigger than $\frac{1}{3}$. So, random guessing would hurt the contestant, but if (for example) he can eliminate two of the answers, it makes sense for him to guess.

At the level $x = 125000$, our computations are similar. This time we have to solve the inequality

$$32000 + \delta \cdot (250000 - 32000) > 125000,$$

which is equivalent to $\delta > \frac{93}{218}$. This is bigger, which makes sense: proportionally he is risking more – he would go down two levels, rather than just one.

Of course, working with only one question at a time is a little bit misleading. For example, consider the $125,000 question. Even after phoning a friend (and using the last of his lifelines), he has no idea. If he will only go one more question, it is clearly correct to walk, but *what if the $250,000 question is something he definitely knows?*

Let us go one step further in our analysis (and you can see how to do still better). Suppose that the contestant estimates that there is a 40% chance that the $250,000 question is one he will know the answer to. If he does, he will guess it correctly, quit the next turn, and walk away with $500,000. If he doesn't, he won't venture a guess and will walk away with $500,000.

In this case, reaching the \$250,000 level is worth

$$0.4 \times 500000 + 0.6 \times 250000 = 350000.$$

So the contestant is risking \$93,000 to win another \$225,000. The expected value of guessing is

$$(1-\delta) \cdot 32000 + \delta \cdot 350000 = 32000 + \delta \cdot 318000,$$

and our inequality is

$$32000 + 318000\delta > 125000,$$

which is equivalent to $\delta > \frac{93}{318}$. In this case it still doesn't make sense for him to randomly guess, but if his guess is even slightly better than random it does. (Moreover, if the contestant estimates that there is a small chance that he would know the answer to the \$500,000 question, this would mean that even a random guess was called for.)

3.2 Linearity of expectation

Example 3.6 *You roll two dice and win a dollar amount equal to the sum of your die rolls. Compute the expected value of this game.*

Solution. (**Hard Solution**). The possible outcomes and the probabilities of each are listed in the table below.

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The expected value is therefore

$$2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}$$
$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = \frac{252}{36} = 7,$$

or exactly 7 dollars.

You should always be suspicious when you do a messy computation and get a simple result.

Solution. (**Easy Solution**). If you roll one die and get the dollar amount showing, we already computed that the expected value of this game is 3.5.

The game discussed now is equivalent to playing this game twice. So the expected value is $3.5 \times 2 = 7$.

Similarly, the expected value of throwing a thousand dice and winning a dollar amount equal to the number of pips showing is (exactly) \$3,500.

Here is another problem that illustrates the same principle.

Example 3.7 *Consider once again the game of Rat Race. Suppose that our contestant gets to pick two out of five rats, that first place wins a car (worth $16,000), that second place wins meal service (worth $2,000) and that third place wins a guitar (worth $500).*

The hard solution would be to compute the probability of every possible outcome: the contestant wins the car and the meals, the car and the guitar, the guitar and the meals, the car only, the meals only, the guitar only, and nothing. What a mess!!! Instead, we'll give an easier solution.

Solution. Consider only the first of the contestant's rats. Since this rat will win each of the three prizes for the contestant with probability $\frac{1}{5}$, the expected value of this rat's winnings is

$$16000 \times \frac{1}{5} + 2000 \times \frac{1}{5} + 500 \times \frac{1}{5} = 3700.$$

The second rat is subject to the same rules, so the expected value of its winnings is also \$3700. Therefore, the total expected value is $\$3,700 + \$3,700 = \$7,400$.

Indeed, the expected value of the game is $3,700 per rat won, so this computation gives the answer no matter how many rats she wins.

There is a subtlety going on in this example, which is noteworthy because we **didn't** worry about it. Suppose, for example, that the first rat fails to even move from the starting line. It is a colossal zonk for the contestant, who must pin all of her hopes on her one remaining rat. Does this mean that her expected value plummets to $3,700? *No!* It now has a one in *four* chance of winning each of the three remaining prizes, so its expected value is now

$$16000 \times \frac{1}{4} + 2000 \times \frac{1}{4} + 500 \times \frac{1}{4} = 4625.$$

Conversely, suppose that this rat races out from the starting block like Usain Bolt, and wins the car! Then the expected value of the remaining rat goes *down.* (It has to: the car is off the table, and the most it can win is $2,000.) Its expected value is a measly

$$2000 \times \frac{1}{4} + 500 \times \frac{1}{4} = 625.$$

This looks terribly complicated, because **the outcomes of the two rats** are not independent. If the first rat does poorly, the second rat is more likely to do well, and vice versa.

The principle of **linearity of expectation** says that our previous computation **is correct, even though the outcomes are not independent.** If the first rat wins the car, the second rat's expected value goes down; if the first rat loses or wins a small prize, the second rat's expected value goes up; and these possibilities average out.

Principle of Linearity of Expectation. Suppose that we have a random process which can be broken up into two or more separate processes. Then, the total expected value is equal to the sum of the expected values of the smaller processes.

This is true whether or not the smaller processes are independent of each other.

Often, games can be broken up in multiple ways. In the exercises you will redo the Rat Race computation a different way: you will consider the expected value of winning just the car, just the meals, and just the guitar – and you will verify that you again get the same answer.

We can now compute the expected value of Rat Race as a whole! Recall that Rat Race begins with the contestant attempting to price three small items correctly, and winning one rat for each item that she gets right.

Example 3.8 *Assume for each small item, the contestant has a 50-50 chance of pricing it correctly. Compute the expected value of playing of Rat Race.*

Solution. Recall from your homework exercises that the probability of winning zero, one, two, or three rats is $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$. Since the expected value of Rat Race is \$3.700 per rat won, the expected value of the race is respectively \$0, \$3,700, \$7,400, and \$11,100. Therefore the expected value of Rat Race is

$$0 \times \frac{1}{8} + 3700 \times \frac{3}{8} + 7400 \times \frac{3}{8} + 11000 \times \frac{3}{8} = 5550.$$

This solution is perfectly correct, but it misses a shortcut. We can use linearity of expectation again!

Solution. Each attempt to win a small item has probability $\frac{1}{2}$ of winning a rat, which contributes \$3,700 to the expected value. Therefore the expected value of each attempt is $3700 \times \frac{1}{2} = 1850$. By linearity of expectation, the expected value of three attempts is

$$1850 + 1850 + 1850 = 5550.$$

3.3 A further expected value example

The St. Petersburg Paradox. You play a game as follows. You start with \$2, and you play the following game. You flip a coin. If it comes up tails, then you win the \$2. If it comes up heads, then your stake is doubled and you get to flip again. You keep flipping the coin, and doubling the stake for every flip of heads, until eventually you flip tails and the game ends.

How much should you be willing to pay to play this game?

To say the same thing another way, your winnings depend on the number of consecutive heads you flip. If none, you win \$2; if one, you win \$4; if two, you win \$8, and so on. More generally, if you flip k consecutive heads before flipping tails, you win 2^{k+1} dollars. Unlike most game shows, you never risk anything and so you will certainly continue flipping until you flip tails.

We first compute the probability of every possible outcome:

- With probability $\frac{1}{2}$, you flip tails on the first flip and win \$2.
- With probability $\frac{1}{4}$, you flip heads on the first flip and tails on the second flip: the probability for each is $\frac{1}{2}$ and you multiply them. If this happens, you win \$4.
- With probability $\frac{1}{8}$, you flip heads on the first two flips and tails on the third flip: the probability for each is $\frac{1}{2}$ so the probability is $\left(\frac{1}{2}\right)^3$. If this happens, you win \$8.
- Now, we'll handle all the remaining cases at once. Let k be the number of consecutive heads you flip before flipping a tail. Then, the probability of this outcome is $\left(\frac{1}{2}\right)^{k+1}$: we've made $k+1$ flips and specified the result for each of them.

  Your winnings will be 2^{k+1} dollars: you start with \$2, and you double your winnings for each of the heads you flipped.

We now compute the expected value of this game. This time there are infinitely many possible outcomes, but we do the computation in the same way. We multiply the probabilities by the expected winnings above, and add:

$$\$2 \cdot \frac{1}{2} + \$4 \cdot \frac{1}{4} + \$8 \cdot \frac{1}{8} + \$16 \cdot \frac{1}{16} + \cdots = \$1 + \$1 + \$1 + \$1 + \cdots = \infty$$

The expected value of the game is infinite, and you should be willing to pay an infinite amount of money to play it. This does not seem to make sense.

By contrast, consider the following version of the game. It has the same rules, only the game has a maximum of 100 flips. If you flip 100 heads, then you don't get to keep playing, and you're forced to settle for 2^{101} dollars, that is, \$2,535,301,200,456,458,802,993,406,410,752.

The expected value of *this* game is a mere

$$\$2\cdot\frac{1}{2}+\$4\cdot\frac{1}{4}+\$8\cdot\frac{1}{8}+\$16\cdot\frac{1}{16}+\cdots+\$2^{100}\cdot\frac{1}{2^{100}}+\$2^{101}\cdot\frac{1}{2^{100}} = \$1+\$1+\$1+\$1+\cdots+\$1+\$2 = \$102.$$

Now think about it. If you won the maximum prize, and it was offered to you in \$100 bills, it would weigh[3] 2.5×10^{25} kilograms, in comparison to the weight of the earth which is only 6×10^{24} kilograms. If you stacked them, you could reach any object which has been observed anywhere in the universe.

Conversely, suppose that it were offered to you in \$100,000,000,000,000 (100 trillion) dollar bills[4] This is much more realistic to imagine; it's roughly equivalent to the Himalayan mountain range being made of such banknotes, all of which belong to you. If you went out to lunch, you'd probably leave a generous tip. You'd have to, because it's not like they can make change for you.

In summary: **This is obviously ridiculous.** You can read more on the Wikipedia article, but the point is that *the real-life meaning of expected values can be distorted by extremely large, and extremely improbable, events.*

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[3] more precisely: have a mass of

[4] Such banknotes were actually printed in Zimbabwe. See, for example, `https://en.wikipedia.org/wiki/Zimbabwean_dollar`.

3.4 Exercises

1. Watch the Deal or No Deal clip from the introduction. Fast forward through all the talk and choosing briefcases if you like, but pay attention to each time the bank offers him a buyout to quit. Compute, in each case, the expected value of playing the game out until the end. Does the bank ever offer a payout larger than the expected value?

   What would you decide at each stage? Explain.

2. Consider again a game of Rat Race with two rats, played for prizes worth \$16,000 (car), \$2,000 (meals), and \$500 (guitar).

   (a) Compute the expected value of the game, considering only the car and ignoring the other prizes. (This should be easy: she has a 2 in 5 chance of winning the car.)

   Solution. She has a $\frac{2}{5}$ chance of winning the car, so the answer is $\frac{2}{5} \times 16000 = 6400$.

   (b) Compute the expected value of the game, considering only the meals.

   Solution. As above, the answer is $\frac{2}{5} \times 2000 = 800$.

   (c) Compute the expected value of the game, considering only the guitar.

   Solution. As above, the answer is $\frac{2}{5} \times 500 = 200$.

   (d) By linearity of expectation, the expected value of the game is equal to the sum of the three expected values you just computed. Verify that this sum is equal to \$7,400, as we computed before.

   Solution. $6400 + 800 + 200 = 7400$.

The next questions concern the Price is Right game **Let 'em Roll**. Here is a clip:

`https://www.youtube.com/watch?v=g5qF-W9cSpo`

Game Description (Let 'em Roll (Price Is Right)):

The contestant has five dice to roll. Each die has \$500 on one side, \$1,000 on another, \$1,500 on a third, and a car symbol on the other three. The contestant rolls all five dice. If a car symbol is showing on each of them, she wins the car. Otherwise, she wins the total amount of money showing. (Car symbols count nothing, unless she wins the car.)

By default, the contestant gets one roll, and may earn up to two more by correctly pricing small grocery items. After each roll, if she gets another roll, she may either keep all the money showing, or set the dice showing 'car' aside and reroll only the rest.

3. First, consider a game of Let ’em Roll where the contestant only gets one dice roll.

   (a) Compute the probability that she wins the car.

   (b) Compute the expected value of the game, considering the car and ignoring the money. (The announcer says that the car is worth $16,570.)

   (c) Compute the expected value of the game, considering the money and ignoring the car.

   (d) Compute the total expected value of the game.

   Solution. The probability that she wins the car is $(\frac{1}{2})^5 = \frac{1}{32}$: there are five dice, and each must show a car.

   Considering only the car, the expected value of the game is $\frac{1}{32} \times 16570 \sim \$518$.

   Considering only the money, each die contributes an expected value of

$$\frac{1}{6} \times 500 + \frac{1}{6} \times 1000 + \frac{1}{6} \times 1500 = 500.$$

   Since there are five dice, the total is $2500, and the total (including both car and dice) is $3018.

4. (a) Now watch the contestant’s playing of the game, where after the second round she chooses to give up $2,500 and reroll. Compute the expected value of doing so. Do you agree with her decision?

   (b) Suppose that after two turns she had rolled no car symbols, and $1,500 was showing on each of the five dice. Compute the expected value of rerolling, and explain why she should *not* reroll.

   (c) Construct a hypothetical situation where the expected value of rerolling is within $500 of not rerolling, so that the decision to reroll is nearly a tossup.

   Solution. After her second round, she has three cars (which she would keep if she rerolls) and $2,500. If she rerolls, she has a one in four probability of winning the car, so her expected value from the car is $\frac{1}{4} \times 16570 \sim 4142$. She also obtains an additional expected value of $1000 from the money, for a total of $5142. As this is much larger than $2,500, rerolling is a good idea if she can stomach some risk.

   In the second scenario, the expected value is the same as the one-turn version (because she will reroll everything): $3,018. Since this is much less than $7,500, it is a good idea to keep the money.

   Here is an intermediate scenario. Suppose two cars are showing and she rerolls the other three dice. Then the expected value of the game is

$$\frac{1}{8} \times 16570 + 3 \times 500 \sim 3571.$$

So if the three money dice are showing a total of \$3,500, it is essentially a tossup decision whether or not to reroll.

As another correct solution, suppose only one car is showing and she rerolls the other four.

$$\frac{1}{16} \times 16570 + 4 \times 500 \sim 3035.$$

If the four money dice are showing \$3000 total, once again it is approximately a tossup.

Yet another correct solution has no cars showing and low amounts of money on the dice: a total of either \$2500 or \$3000.

5. If the contestant prices the small grocery items correctly and plays optimally, compute the expected value of a game of Let 'em Roll.

   (Warning: if your solution is simple, then it's wrong.)

4 Counting

We now consider a variety of clever counting methods, which will be useful in sophisticated probability computations.

4.1 The Multiplication Rule

Just as there was a multiplication rule for probability, there is a multiplication rule for counting as well. It is as follows.[5]

The multiplication rule for counting. Suppose that an operation consists of k steps, and:

- The first step can be performed in n_1 ways;
- The second step can be performed in n_2 ways (regardless of how the first step was performed);
- and so on. Finally the kth step can be performed in n_k ways (regardless of how the preceding steps were performed).

Then the entire operation can be performed in $n_1 n_2 \dots n_k$ ways.

Example 4.1 *In South Carolina, a license tag can consist of any three letters followed by any three numbers. (Example: TPQ-909) How many different license tags are possible?*

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[5] We adopt the wording of Epp, *Discrete Mathematics with Applications*, 4th ed., p. 527.

Solution. There are 26 possibilities for the first letter, 26 for the second, and 26 for the third. Similarly there are 10 possibilities for each number. So the total number of possibilities is $26^3 \cdot 10^3 = 17576000$.

Note that big states with more people than South Carolina have started using different license plate schemes, because they ran out of possible tags.

Example 4.2 *How many license tags are possible which don't repeat any letters or numbers?*

Solution. There are still 26 possibilities for the first letter, and now 25 for the second and 24 for the third: we must avoid the letters that were previously used. Similarly there are 10, 9, and 8 possibilities for the three numbers. The total number of possibilities is

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11232000.$$

These computations may be used to solve probability questions. For example:

Example 4.3 *What is the probability that a random license tag doesn't repeat any letters or numbers?*

This follows from the previous two computations. The result is

$$\frac{11232000}{17576000} = .639\ldots$$

Example 4.4 *On a game of Ten Chances, Drew Carey feels particularly sadistic and puts all ten digits – zero through nine – to choose from in the price of the car. The price of the car consists of five different digits. How many possibilities are there?*

Solution. There are 10 possibilities for the first digit, 9 for the second, 8 for the third, 7 for the fourth, and 6 for the fifth, for a total of

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30240$$

possibilities. Good luck to the poor sucker playing this game.

Example 4.5 *As above, but suppose you know that the first digit is not zero. Now how many possibilities are there?*

Solution. This time there are only 9 possibilities for the first digit. There are still 9 possibilities for the second, no matter what the first digit was, and 8, 7, 6 for the last three in turn. The total is

$$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216.$$

Example 4.6 *As above, but suppose you know that the first digit is not zero* **and** *that the last digit is zero. Now how many possibilities are there?*

Solution. There are 9 possibilties for the first digit, 9 for the second, 8 for the third, 7 for the fourth, and ... **either** 0 **or** 1 **for the last depending on whether we've used the zero.** No good! We can't use the multiplication rule this way!

To use the multiplication rule, we pick the numbers in a different order: the first digit first (anything other than the zero, 9 ways), then the last digit (must be the zero, so 1 way), and then the second, third, and fourth digits in turn (8, 7, and 6 ways), for a total of

$$9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

ways.

Alternatively, we could have picked the last digit before the first, and we can pick the second, third, and fourth digits in any order. It is usually best to find one order which works and stick to it.

4.2 Permutations and combinations

Recall that a **permutation** of a **string with** n **symbols** is any reordering of the string. If the symbols are all distinct, then there are $n!$ possible permutations of it. We justified this earlier, and it is an example of the multiplication rule: there are n ways to choose the first symbol, $n-1$ to choose the second, $n-2$ to choose the third, and so on.

Implicitly, we also discussed what are called r-permutations. If $r \leq n$, then an r-permutation of a string of length n is a reordering of r of the n symbols. For example, 16820, 98561, and 37682 are 5-permutations of the string 1234567890. We discussed these in our Ten Chances examples above, and there are 30240.

Notation. Write $P(n, r)$ for the number of r-permutations of a string with n distinct symbols.

We have the following formula:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Why is this true? It comes from the multiplication rule. There are n possibilities for the first symbol, $n-1$ possibilities for the second, and so on: one less for each subsequent symbol. There are $n-r+1$ possibilities for the rth symbol: we start at n and count down by 1 $r-1$ times. So we see that

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots (n-r+1).$$

Why is this equal to $\frac{n!}{(n-r)!}$? Our expression is the same as $n!$, except that the numbers from $n-r$ down to 1 are all absent. So we've left out a product equalling $(n-r)!$ from the definition of $n!$, and so it equals $\frac{n!}{(n-r)!}$.

Example 4.7 *Compute $P(n, r)$ for all possible n and r with $r \leq n \leq 6$ and notice any patterns.*

Solution. (To be written up here)

Combinations. Combinations are like permutations, only the order doesn't matter.

If we start with a string (or a set) with n distinct elements, then an r**-combination** is a string or r of these elements *where order doesn't matter*, or equivalently a subset of r of these elements.

Example 4.8 *Write out all the* 3*-combinations of* 12345*.*

Solution. They are: 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345. There are ten of them.

Here, we could have equivalently written 321, 213, or $\{1, 2, 3\}$ (for example) in place of 123, because when counting **combinations** it is irrelevant which order the symbols come in. When counting permutations it **is** relevant, so **please always be careful to pay attention to exactly what you are counting!**

Note that a string with n distinct elements, where order doesn't matter, is the same thing as a set of n distinct elements. We won't worry about distinguishing these too carefully, although in advanced mathematics and in computer programming it is important to be precise.

Example 4.9 *Write out all the* 2*-combinations of* 12345*.*

Solution. They are: 45, 35, 34, 25, 24, 23, 15, 14, 13, and 12. Again, there are ten of them.

I didn't have to list them in reverse order, but in doing so we notice something interesting: they correspond exactly to the 3-combinations! Choosing which two elements to include is equivalent to choosing which three to leave out, so we can line up the list of 2-combinations with the list of 3-combinations and see that there is a one-to-one correspondence. In mathematical parlance, we call this a **bijection.** *If you want to prove that two sets have the same size, finding a bijection between them is a great way to do it!*

Notation. Write $C(n, r)$ or $\binom{n}{r}$ for the number of r-combinations of an n-element set.

The latter notation is read "n choose r", and is ubiquitous in mathematics. These numbers are also called 'binomial coefficients', because we have

$$(x+1)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \cdots + \binom{n}{1}x + \binom{n}{0}.$$

For example, we have

$$(x+1)^10 = x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1,$$

so we can FOIL without FOILing. If you think about it carefully, you can figure out why the first equation is true. But we still haven't explained how to actually *compute* these things. Here's the answer.

Theorem 4.10 *We have*

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

To explain thiis we will be very careful and work backwards. First of all, note that it is enough to show that

$$P(n, r) = C(n, r) \cdot r! \tag{2}$$

We will first explain why (4.2) implies the theorem, and then we will explain why (4.2) is true. First, note that (4.2) implies that

$$C(n, r) = \frac{P(n, r)}{r!},$$

but remember that we showed that $P(n, r) = \frac{n!}{(n-r)!}$. Therefore,

$$C(n, r) = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!r!},$$

as desired.

We are left to explain why is true. To do this, we explain why both sides of (4.2) count the number of r-permutations of a string of n elements:

- This is true of $P(n, r)$ by definition.
- Instead, we could first choose which r objects to make an r-permutation out of, without worrying about the order. By definition, there are $C(n, r)$ ways to do this. Now, we have to put these r symbols in some order – i.e., to write down a permutation of them. There are $r!$ ways to do this. So the total number of ways is $C(n, r) \cdot r!$

If you haven't seen this before, you probably didn't understand what just happened. That's okay. Read through it again.

4.3 Pascal's Triangle

The part of this section with the diagrams has been removed from this version of the notes. It will again be in the final version – I removed it here because it is much faster to edit the file if I don't include the diagrams. Please see (for example) the September 30 version of the file to see this part of these notes.

Here is a video of the Price Is Right game **Plinko**:

`https://www.youtube.com/watch?v=qr7oYqcgsXQ`

Game Description (Plinko (The Price Is Right)): The contestant drops up to five chips down a board. (She starts off with one, and can win up to four more by pricing small items.) She drops them down a board which has a lot of pegs and a variety of prizes at the bottom. (The shape of the board **is** relevant, and we will discuss it more in due course.) She hopes to land her chips into a $10,000 slot in the middle, and the other slots have prizes between zero and $1,000.

The question is, **where should the contestant drop her pucks**?

Here is a graphical representation of a Plinko board.

[Temporarily removed.]

For comparison's sake we have included the previous numbers (where the board had walls) below the final numbers. They are different, but not *so* different.
We have just written out the first thirteen rows of **Pascal's Triangle**.

Pascal's Triangle. To write down Pascal's Triangle, proceed as follows.

- The top row has a solitary 1 in it.
- Each row has one more number than the previous, with a 1 at each edge. **Each number in the middle of the table is equal to the sum of the two above it.**
- Proceed for as many rows as you like.
- By convention the rows are numbered as follows: the top row is the **zeroth** row. After that, the rows are numbered 1, 2, 3, etc., and the nth row starts with a 1 and an n.

Our idealized version of Plinko is illustrated nicely by the following computer demonstration:

`phet.colorado.edu/sims/plinko-probability/plinko-probability_en.html`

We now investigate Pascal's Triangle and observe that it has a large number of remarkable properties:

Proposition 4.11 *The numbers in the nth row of Pascal's Triangle sum to* 2^n.

Why is this? It is true for the 1st row, and we see that each number contributes twice to the row below it: once to its left, and once to its right. Hence, the sum of each row is twice that of the row above it.

Proposition 4.12 *The numbers in the nth row of Pascal's Triangle are* $C(n,0)$, $C(n,1)$, *...,* $C(n,n)$ *in order.*

This is remarkable! Note that this lets us compute $C(n, r)$ for all r without any factorials.

Why is this true? Consider, as an example, the 15 sitting left of center in the sixth row. This 15 counts the number of possible paths a ball could have taken from the top 1 to this 15.

How many such paths are there? We already saw that the answer is 15, but let's see the same thing another way. The ball took six steps down, and of these four were to the left and two were to the right. Say you label these steps A, B, C, D, E, and F in order. Then, for each choice of two of these six letters (note: there are $C(6, 2)$ such choices), there is exactly one path corresponding to that choice. For example, corresponding to the choice $\{B, D\}$ is the path Left, Right, Left, Right, Left, Left.

The reasoning is the same no matter which number we started with.

Proposition 4.13 *We have $C(n, r) = C(n, n - r)$ for all n and r.*

This can be seen from the symmetry in the triangle. It can also be seen as a consequence of our explicit formula. We have

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

and

$$C(n, n - r) = \frac{n!}{(n - r)!(n - (n - r))!}$$

But $n - (n - r)$ is just the same thing as r.

Proposition 4.14 *The biggest numbers are always in the middle.*

If it is true for one row, it has to be true for the row below it (since this row is made of sums of the row above it). So since it is true at the top, it must always be true.

Note that this tells you how you should drop the puck in Plinko: *drop it directly down the middle, right over the \$10,000!*

Proposition 4.15 *We have, for all n and r, that*

$$C(n, r) + C(n, r + 1) = C(n + 1, r + 1).$$

This is just a restatement of how Pascal's Triangle is constructed. Since the rows consist of the quantities $C(n, r)$, we get the above identity.

Note that algebraically, this says that

$$\frac{n!}{r!(n - r)!} + \frac{n!}{(r + 1)!(n - (r + 1))!} = \frac{(n + 1)!}{(r + 1)!((n + 1) - (r + 1))!},$$

which after simplifying a bit is the same thing as

$$\frac{n!}{r!(n-r)!}+\frac{n!}{(r+1)!(n-r-1)!}=\frac{(n+1)!}{(r+1)!(n-r)!}, \tag{3}$$

We can also verify this algebraically. A common denominator for the fractions in (4) is $(r+1)!(n-r)!$, so the left side of (4) is

(4)

$$\frac{n!(r+1)}{(r+1)!(n-r)!}+\frac{n!(n-r)}{(r+1)!(n-r)!}=\frac{n!(r+1+n-r)}{(r+1)!(n-r)!}=\frac{n!(n+1)}{(r+1)!(n-r)!}=\frac{(n+1)!}{(r+1)!(n-r)!}.$$

But that was a lot less fun.

Proposition 4.16 *You can read off a rule for FOILing from Pascal's Triangle. In particular, you have*

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \cdots + C(n,n)y^n.$$

This is called the **binomial theorem**.

For example, plug in $x=1$ and $y=1$. You get

$$2^n = C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n).$$

In other words, we see again that the sum of the nth row is 2^n.

Proposition 4.17 *The alternating sum of each row of Pascal's Triangle (after the zeroth) is* 0.

For example, the alternating sum of the seventh row is

$$1-7+21-35+35-21+7-1=0.$$

Here, you can tell immediately that these sum to zero, because the numbers cancel in pairs. But look at the eighth row. We also have

$$1-8+28-56+70-56+28-8+1=0.$$

This is not obvious immediately — unless you look at the binomial theorem! Plug in $x=1$ and $y=-1$. We get

$$0 = C(n,0) - C(n,1) + C(n,2) - C(n,3) + \cdots \pm C(n,n).$$

The last ± is a plus if n is even and a minus if n is odd.

Proposition 4.18 *If you color all the odd numbers blue and the even numbers red, you will create a familiar pattern called the 'Sierpinski triangle' which is a* **fractal**.

Try it!!

Proposition 4.19 *Suppose you draw lines through Pascal's Triangle at an angle.*

For example, start at any of the 1's on the left. Circle it. Then, go over to the right one and up and right one, and circle that number. Then, again go over to the right one and up and right one and circle that. Keep going until you run out of numbers.

If you add up all the numbers you circled, you get .....

What, do you expect me to do it for you? And spoil the surprise? No way. Try it!! (Try multiple such lines, and see if you can find the pattern.)

Proposition 4.20 *The distribution of Pascal's triangle approaches a nice limit as $n \to \infty$.*

This is subtle, and so our explanation here will be a bit vague. Consider the following question. You flip a fair coin a million times. What is the probability that you get at least 501,000 heads?

We can give the answer immediately: it is

$$2^{-1000000}\bigg(C(1000000, 501000)+C(1000000, 501001)+C(1000000, 501002)+\cdots C(1000000, 1000000)\bigg).$$

But in some sense this is a useless answer. If I asked you whether this was nearer to 20% or 0.0000000000000001%, could you answer just by looking at it? (No.)

Here is a website which allows you to conduct experiments like this:

`http://www.math.uah.edu/stat/apps/BinomialTimelineExperiment.html`

The variable n is the number of coin flips and you do that many flips, many times over. So you can see graphically the answers to questions like this. What it is important to observe is that *the shape of the graph is, in some sense independent of n.*

The limiting distribution is called a *normal* or *Gaussian* distribution, or more informally *the bell curve.* It has *mean* (i.e., average) $\frac{n}{2}$ and *standard deviation* $\frac{\sqrt{n}}{2}$. Roughly speaking, the *standard deviation* is a measure of how much you might reasonably expect a trial to be off from the mean.

So, for example, if you flip 1,000,000 coins, the mean outcome is 500,000 heads, the standard deviation is 500, and 501,000 is two standard deviations away. The probability of this outcome is roughly 2.27% – unlikely, but you wouldn't be shocked to see it happen. Conversely, the probability of getting at least 503,000 heads is less than a billion.

4.4 Exercises

Incomplete. To be added to.

1. Compute tables of $P(n,r)$ and $C(n,r)$ for all n and r with $0 \leq r \leq n \leq 8$.

2. Explain why $C(n,0) = 1$ and $C(n,1) = n$ for all n. Can you explain this using the definition instead of the formula?

3. The following clip is from the game show **Scrabble**:

   `https://www.youtube.com/watch?v=iliCKnHxJiQ`

   (a) At 6:25 in the video, Michael chooses two from eleven numbered tiles. The order in which he chooses them doesn't matter. Eight of the tiles are 'good', and reveal letters which are actually in the word. Three of them are 'stoppers'.

   How many different choices can he make?

   Solution. $C(11,2) = 55$.

   In this example, it turns out that there are two R tiles, and two D tiles (one of which is a stopper). The easy solution presumes that these are different from each other – that one of the numbered tiles is the R appearing first in the word, and another one is the R appearing second in the word.

   However, if you watch the show a lot you will observe this is *not actually true* – the *first D* picked will always be the good one, and the second will always be the stopper. Our solution is the 'easy solution' – extra credit to anyone who observed that this is not quite accurate, and took account of it!

   (b) Of these choices, how many choices don't contain a stopper? If he places both letters, what is the probability that both will actually appear in the word?

   Solution. $C(8,2) = 28$, and so $\frac{28}{55}$.

   (c) Michael can't guess the word and chooses two more of the remaining tiles. Now what is the probability that both of them will actually appear in the word?

   Solution. Now there are nine remaining tiles and six of them are good. It's $\frac{C(6,2)}{C(9,2)} = \frac{15}{36} = \frac{5}{18}$. Not very good.

   (d) At 8:15 (and for a different word), the contestants have used up two of the stoppers. Now what is the probability that both of Michael's letters will appear in the word?

   Solution. There are six letters, and he chooses two. The probability that neither is the bad one is $\frac{4}{6} = \frac{2}{3}$.

   (e) (Challenge!) Suppose that Michael knows the first (6:25) word from the beginning, but rather than guessing it immediately chooses to draw tiles until one of the following happens: (1) he draws and places the first R on the blue spot, and thereby can earn \$500 for his guess; (2) he draws two stoppers, and must play one of them (and so forfeits his turn); (3) he places all letters but the first R, and is obliged to guess without earning the \$500.

Compute the probabilities of each of these outcomes.

Solution. We look at this turn by turn.

- (First turn.) With probability $\frac{3}{55}$ he draws two stoppers and loses. With probability $\frac{10}{55} = \frac{2}{11}$ he draws the first R and can place it and win \$500.
  The number of ways in which he can draw one stopper and one good tile other than the first R is $3 \cdot 7 = 21$, so there is probability $\frac{21}{55}$ that this happens and he wins the turn but not \$500. Finally, there are $C(7,2) = 21$ in which he can draw good two tiles other than the first R, so there is probability $\frac{21}{55}$ that this will happen and he goes to a second round.
  Note that $3 + 10 + 21 + 21 = 55$ – a good way to check our work! We've enumerated all possibilities and the probabilities end up to 1.
- (Second turn.) There is probability $\frac{21}{55}$ that the game goes on to a second turn. The following probabilities assume that it does, and should all be multiplied by $\frac{21}{55}$.
  There are $C(9,2) = 36$ ways to draw two tiles. As above, there are 3 ways to draw two stoppers, 8 ways in which he can draw the first R and something else, $3 \cdot 5 = 15$ ways in which he can draw a stopper and a tile other than the first R, and $C(5,2) = 10$ ways in which he can draw two more good tiles other than the first R. So there is probability $\frac{10}{36}$, or $\frac{21}{55} \times \frac{10}{36}$ total, of the game going onto a third round.
- (Third turn.) Similar to above. There are $C(7,2) = 21$ ways to draw two tiles, 3 to draw two stoppers, 6 to draw the first R, 9 to draw a stopper and a tile other than the first R, and 3 ways in which he can draw two more good tiles other than the first R. The probability of the game going on to a fourth turn (total) is $\frac{21}{55} \times \frac{10}{36} \times \frac{3}{21}$.
- (Fourth turn.) There are $C(5,2) = 10$ ways to draw two tiles, 3 to draw two stoppers, 4 to draw the first R, and 3 ways in which he can draw a stopper and a title other than the first R.

So we can compute all the probabilities:

- Places the first R:
$$\frac{10}{55} + \frac{21}{55} \cdot \frac{8}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{6}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{4}{10} = \frac{10}{33}$$
- Draws two stoppers:
$$\frac{3}{55} + \frac{21}{55} \cdot \frac{3}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{3}{10} = \frac{7}{66}$$
- Must guess without winning \$500:
$$\frac{21}{55} + \frac{21}{55} \cdot \frac{15}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{9}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{3}{10} = \frac{13}{22}.$$

4. Consider our first model of Plinko, where we assumed that the puck would always go one space to the left or one space to the right, but did not ignore the walls of the board.
    (a) If the contestant drops the puck one slot to the left of center, we already computed the probability that the puck lands in each of the nine slots. Compute the expected value of this drop. (Use a calculator or computer, and round to the nearest dollar.)
    (b) Carry out all these computations (1) if the contestant drops the puck down the center, and (2) if the contestant drops the puck down the far left slot. If you have the patience, you might also do it if the contestant drops the puck two left of center – in this case, by symmetry, you will have considered all the possibilities.

        What can you conclude about where the contestant should drop the puck?
5. Watch one or more playings of Plinko, and discuss the shortcomings in our model. Does the puck ever go more than one space to the left or right?

    Briefly discuss how you would revise the model to be more accurate, and summarize how you would redo the problem above to correspond to your revised model. (The details are likely to be messy, so you're welcome to not carry them out.)

5 Example: Poker

We digress from our discussion of 'traditional' game shows to discuss the game of *poker*. Poker is frequently televised – for example you can find the final table of the 2014 World Series of Poker on YouTube, all fourteen hours of it – so you might call it a game show. Poker is a very mathematical game, and we can very much use the mathematics we have already developed to analyze it.

We start off by describing the poker hands from best to worst and solving the combinatorial problems which naturally arise. For example, if you are dealt five cards at random, what is the probability that you get dealt a straight or better?

We will then discuss different variations of Poker and the betting rules. This is where the really interesting decisions come into play: do you fold, call, or raise? These are *essentially* expected value computations, although you must make informed guesses about what your competitors hold.

Online play. There are several websites where you can play free poker on the Internet. One I have used myself is

`http://www.replaypoker.com`

– there is no gambling. (There is *betting*, but you are playing for 'chips' which do not represent real money.)

Further reading. There are a great many excellent books on poker. I especially recommend the *Harrington on Hold'em* series by Dan Harrington. These books are quite sophisticated and walk you through a number of expected value and probability computations. If you've ever wanted to learn to play, you will find that this course provides excellent background!

5.1 Poker Hands

A **poker hand** consists of five playing cards. From best to worst, they are ranked as follows:

- **A straight flush**, five consecutive cards of the same suit, e.g. 5♠6♠7♠8♠9♠. An ace may be counted high or low but straights may not 'wrap around' (e.g. KA234 is not a straight).

  In case of a tie, the high card in the straight flush settles ties. An ace-high straight flush is sometimes called a **royal flush**, and is the highest hand in poker.

- **Four of a kind**, for example $K\spadesuit K\clubsuit K\diamondsuit K\heartsuit$ and any other card. (If two players have four of a kind, the highest set of four cards win.)

- **A full house**, i.e. three of a kind and a pair, $K\spadesuit K\clubsuit K\diamondsuit 7\heartsuit 7\diamondsuit$. (If two players have a full house, the highest set of three cards wins.)

- **A flush**, any five cards of the same suit, e.g. $Q\clubsuit 10\clubsuit 7\clubsuit 6\clubsuit 3\clubsuit$. The high card breaks ties (followed by the second highest, etc.)

- **A straight**, any five consecutive cards, e.g. 8♣7♢6♢5♡4♠. The high card breaks ties.

- **Three of a kind**, e.g. $8\clubsuit 8\diamondsuit 8\spadesuit A\heartsuit 4\spadesuit$.

- **Two pair**, e.g. $8\clubsuit 8\diamondsuit 6\spadesuit 6\heartsuit A\spadesuit$.

- **One pair**, e.g. $8\clubsuit 8\diamondsuit 6\spadesuit 5\heartsuit A\spadesuit$.

- **High card**, e.g. none of the above. The value of your hand is determined by the highest card in it; then, ties are settled by the second highest card, and so on.

We now compute *the probability of each possible hand occurring.* Our computations will make heavy use of the multiplication rule. (Note that each card is determined uniquely by its *rank* (e.g. king, six) and *suit* (e.g., spades, clubs).)

- **All hands**. The total number of possible hands is $C(52, 5) = 2598960$.

- **Straight flush** (including royal flush). There are four possible suits, and nine possible top cards of that suit: ace down through five. These determine the rest of the straight flush, so the total number of possibilities is $4 \times 10 = 40$.

- **Four of a kind**. There are thirteen possible ranks. You must hold all four cards of that suit, and then one of the other 48 cards in the deck, so the total number of possibilities is $13 \times 48 = 624$.

- **Full house**. First, choose the rank in which you have three of a kind. There are 13 possible ranks, and $C(4,3) = 4$ choices of three of that rank. Then, choose another rank (12 choices) and two cards ($C(4,2) = 6$) of that rank. The total number of possibilities is the product of all these numbers: $13 \times 4 \times 12 \times 6 = 3744$.

- **Flush**. Choose one of four suits (in 4 ways), and five cards of that suit (in $C(13,5)$ ways), for a total of $4 \times C(13,5) = 5148$ possibilities.

  Except, we don't want to count the straight flushes again! So subtract 40 to get 5108.

- **Straight**. Choose the highest card (ace through five, so ten possibilities). For each of five ranks in the straight, there are 4 cards of that rank, so the number of possibilities is $10 \times 4^5 = 10240$. Again subtracting off the straight flushes, we get 10200.

- **Three of a kind.** Choose a rank and three cards of that rank in $13 \times C(4,3) = 52$ ways. Then, choose two other ranks (distinct from each other) in $C(12,2)$ ways. For each of these ranks there are four possibilities, so the total is $52 \times C(12,2) \times 4^2 = 54912$.

  Note that hands with four of a kind or a full house 'include three of a kind', but we counted so as to exclude these possibilities, so we don't need to subtract them now.

- **Two pair.** Choose two different ranks in $C(13,2)$ ways; for each, choose two cards of that rank in $C(4,2)$ ways. Finally, choose one of the 44 cards not of the two ranks you chose. The total number of possibilities is $C(13,2) \times C(4,2)^2 \times 44 = 123552$.

- **One pair.** Choose the rank in 13 ways and choose two cards of that rank in $C(4,2)$ ways. Then, choose three other ranks in $C(12,3)$ ways and for each choose a card of that rank in 4 ways.

  The total number of possibilities is $13 \times C(4,2) \times C(12,3) \times 4^3 = 1098240$ ways.

- **None of the above.** There are several ways we could count this. Here is one way: we can choose five different ranks in $C(13,5)$ ways – but we must subtract the ten choices that are straights. So the number of choices for ranks is $(C(13,5) - 10)$.

  Now, for each rank, we choose a suit, and the total number of choices is $4^5 - 4$. We subtract 4 because we want to exclude the flushes! So the total number of possibilities is $(C(13,5) - 10) \times (4^5 - 4) = 1302540$.

  Here is a second way to get the same result. We know that the total number of possibilities is 2598960. So we add all the previous possibilities, and subtract from 2598960.

This involved some subtleties, and for other variations the computations are still harder! For example, in **seven card stud** you are dealt a seven-card hand, and you choose your best five cards and make the best possible poker hand from these. You can redo all the above computations, but now some new possibilities emerge. For example, you can be simultaneously dealt a straight and three of a kind – and you want to count this only as a straight (since that is better than three of a kind). But it is not *so* hard. The following Wikipedia page works out all the probabilities in detail:

`https://en.wikipedia.org/wiki/Poker_probability`

Poker variations. There are many variants of poker. The rules for betting (and blinds and antes) are described in the next section; for now we simply indicate when a round of betting occurs.

'Ordinary' poker. (No one actually plays this.) Each player is dealt five cards face down. There is a round of betting. The best hand (among those who have not folded) wins.

Five-card draw. Each player is dealt five cards face down. There is a round of betting. Then, each player who has not folded may choose to trade in up to three cards, which are replaced with new cards (again dealt face down). There is another round of betting, and the best hand wins.

Texas Hold'em. Typically played using blinds (and sometimes also antes), applied to the first round of betting only. Each player is dealt two cards, dealt face down. There is a round of betting. Three community cards are dealt face up (the 'flop'), which every player can use as part of their hand. There is a round of betting. A fourth community card is dealt (the 'turn'), followed by another round of betting. Finally, a fifth community card is dealt (the 'river'), again followed by another round of betting.

Each player (who has not folded) chooses their best possible five-card hand from their two face-down cards and the five face-up cards (the latter of which are shared by all players). The best hand wins.

Texas Hold'em is extremely popular and plenty of video can be found on the internet. For example, this (six hour!) video is of the first part of the final table of the 2014 World Series of Poker:

`https://www.youtube.com/watch?v=5wlVFMNVJZQ`

The top prize was a cool $10 million.

This is the most interesting poker video I have ever seen. Most telecasts of poker heavily edit their coverage, only showing the hands where something exciting or out of the ordinary happens. This video is unedited, and so gives a much more realistic viewpoint of what tournament poker is like.

In the opening round of Texas Hold'em, you are dealt only your two-card hand and you have to bet before any of the community cards are dealt. This offers some probability questions which are quite interesting, and easier than those above. For example, in *Harrington* *on Hold'em, Volume I: Strategic Play*, Harrington gives the following advice for you should raise, assuming you are playing at a full table of nine or ten players and are the first player to act.

- Early (first or second) position: Raise with any pair from aces down to tens, ace-king (suited or unsuited), or ace-queen (suited).
- Middle (third through sixth) position: Raise with the above hands, nines, eights, ace-queen, ace-jack, or king-queen (suited or unsuited).
- Late (seventh or eighth) position: Raise with all the above hands, sevens, ace-x, or high suited connectors like queen-jack or jack-ten.

Harrington also points out that your strategy should depend on your stack size, the other players' stack sizes, your table image, the other players' playing styles, any physical tells you have on the other players, the tournament status, and the phase of the moon. But this is his starting point. Let us work out a few examples (you will be asked to work out more in the exercises).

Example 5.1 *In a game of Texas Hold'em, compute the probability that you are dealt a pair of aces ('pocket aces').*

Solution. There are $C(52,2) = 1326$ possible two-card hands. Of these, $C(4,2) = 6$ are a pair of aces, so the answer is $\frac{6}{1326} = \frac{1}{221}$, a little bit less than 0.5%.

Example 5.2 *In a game of Texas Hold'em, compute the probability that you are dealt a pair.*

Solution. There are 13 possible ranks for a pair, and $C(4,2) = 6$ pairs of each rank, so the answer is $\frac{6 \times 13}{1326} = \frac{1}{17}$.

Example 5.3 *You are playing Texas Hold'em against five opponents, and you are dealt a pair of kings. You have the best hand at the table unless someone else has a pair of aces. Compute the probability that one of your opponents has a pair of aces.*

Approximate solution. There are fifty cards left in the deck, excluding your two kings. The probability that any *specific* one of your opponents has pocket aces is $\frac{C(4,2)}{C(50,2)} = \frac{6}{1225}$, or about 1 in 200. (This much is exact.)

These probabilities are not independent: if one player has pocket aces, the others are less likely to. Nevertheless, we get a very nearly correct answer if we assume they are independent. The probability that any specific player does *not* have pocket aces is $1 - \frac{6}{1225} = \frac{1219}{1225}$. If these probabilities are independent, the probability that all five opponents have something other than pocket aces is $\left(\frac{1219}{1225}\right)^5$. So the probability that at least one of your opponents has pocket aces is

$$1 - \left(\frac{1219}{1225}\right)^5 = 0.0242510680\dots$$

Remark. Here is a simpler approximate solution. Just multiply $\frac{6}{1225}$ by 5, to get

$$\frac{30}{1225} = 0.02448979591\dots$$

This is almost exactly the same. Why is this? We can use the binomial theorem to see that

$$1 - (1 - x)^5 = 5x - 10x^2 + 10x^3 - 5x^4 + x^5,$$

and plug in $x = \frac{6}{1225}$. Since x is very small, the x^2, etc. terms are **very** small.

Example 5.4 *You are sitting in first position. Compute the probability that you receive a hand that you should raise, according to Harrington's advice.*

Solution. As before there are 1326 hands, so we count the various hands that Harrington says are worth opening:

- A pair of aces through tens: Five ranks, and 6 ways to make each pair, so a total of $5 \times 6 = 30$.
- Ace-king: Four ways to choose the suit of the ace, and four ways to choose the suit of the king. $4 \times 4 = 16$.
- Ace-queen suited. (*Suited* means the cards are of the same suit. If your cards are suited, this helps you because it increases the chances that you will make a flush.) Four ways to choose the suit, so just 4.

None of these possibilities overlap, so the total number is $30 + 16 + 4 = 50$. The probability is $\frac{50}{1326}$.

This is less than 1 in 25! Harrington's strategy is much more conservative than that of most top players.

In the exercises, you will compute the probability of getting a hand worth opening in middle or late position.

5.2 Poker Betting

So far we have just considered probabilities. But the interesting part of the game comes when we combine this with a discussion of betting strategy.

Poker is played for *chips*, which may or may not represent money. In general there are two different formats. In a **cash game**, you simply try to win as many chips as you can.

By contrast, a **tournament** is played until one player has won all the chips. Before each hand players have to put **antes** or **blind bets** into the pot, and in a tournament these keep going up and up to force the tournament to end eventually.

Betting rounds. In all variations of poker, a betting round works as follows. The first player (usually, but not always, the player left of the dealer) opens the betting. She may **check** (bet nothing) or bet any amount. The betting then proceeds around the table clockwise. If no one has bet yet, the player may check or bet. If someone has bet, then the player may **fold** (abandon her hand), **call** (match the bet), or **raise** (put in a larger bet). The betting continues to go around the table until either everyone has checked, or everyone has called or folded to the last (largest) bet. Note that players may raise an unlimited number of times, so betting can go around the table multiple times if many players keep raising.

In *no-limit* poker, a player may bet anything up to and including her entire stack of chips. Players are never allowed to bet more than however many chips they have on the table. (You are not allowed to reach into your wallet and suddenly drop a stack of Benjamins.) Conversely, you can always call a bet for your entire stack: if someone bets more chips than you have, you may go 'all-in' and their effective bet is limited to the number of chips you have. (There are 'side pot' rules if one player is all-in and two other players want to keep raising each other; we won't consider them here.)

Typically there are multiple rounds of betting. If a player bets and everyone else folds, then that player wins the pot. (The 'pot' consists of the blinds and antes and all of the bets that have been made.) Otherwise, everyone remaining at the end compares their hands, and the best hand wins the pot.

Blinds and antes. A hand of poker never starts with an empty pot; there is always a little bit of money to be won from the beginning. This is assured via blinds and antes. If **antes** are used, then each player puts a fixed (small) amount of money into the pot at the beginning. If **blinds** are used, then the first two players in the first betting round make a 'blind bet' before looking at their cards. For example, the first player might be required to bet \$1 (the small blind) and the second player \$2 (the big blind). These count as their initial bets, except that if everyone calls or folds to the big blind, the round is not quite over; the big blind has the opportunity to raise if she wishes.

5.3 Examples

We now consider some examples of poker play and the mathematics behind your decision making.

Example 1. You are playing Texas Hold'em with one opponent (Alice). The current pot is 500 chips, and you and Alice each have 700 chips. You have a hand of 5♡4♡, the flop comes A♣K♡10♡. You check, and Alice responds by going all-in. Should you fold or call her bet?

Analysis. There are three steps to solving this problem. First, you estimate your winning probability depending on what cards come. Since you don't know what your opponent has, this is a very inexact science (and indeed depends on your assessment of Alice's strategy).

The next two steps are mathematically more straightforward: the second step is to compute the probability of each possible outcome, and the third is to determine whether the expected value of calling is positive or negative. Since the expected value of folding is always zero (not counting whatever you have put into the pot already), this determines whether or not you should call.

You guess that Alice probably has a good hand – a pair of tens or higher. You estimate that you probably need to make a flush to beat her. You make a flush if at least one heart comes in the turn and the river. You'd rather see *only* one heart, because if two hearts come, Alice beats you if she has any heart higher than the 5♡.

- If exactly one heart comes during the next two cards, then almost certainly you win. You only lose if Alice has two hearts, one of them higher than a five, or if she makes some freak hand like a full house or four of a kind. (This can't be discounted if a pair appears on the flop, but as it stands this looks pretty unlikely.)

  We estimate your winning chances here as 90%. (Reasonable people might disagree!)

- If two hearts come during the next two cards, you might win – but Alice could easily have a heart higher than the 5♡. We estimate your chances of winning as 50%.

- If no hearts come, then you are very unlikely to win. You could – for example, if two fives, or two fours, or a five and a four, come then you *might* win, but this is unlikely. We will simplify by rounding this probability down to zero.

There are 47 cards you can't see, and nine of them are hearts. What is the probability that the next two are both hearts? As we've seen before, this is

$$\frac{9}{47} \cdot \frac{8}{46} \sim 0.033\ldots$$

This is quite low! It is substantially lower than $(1/4)^2$, simply because you can already see four of the hearts.

Now, what is the probability that one, but not both, of the next two cards, is a heart? There are two ways to compute this, and we will work out both.

Method 1. The probability that the first card is a heart and the second card is not a heart is

$$\frac{9}{47} \cdot \frac{38}{46} \sim 0.158\ldots$$

The probability that the second card is a heart and the first card is not is the same. So the total probability is $\frac{342}{1081}$, or approximately 0.316.

Method 2. First, we compute the probability that neither card is a heart. This is

$$\frac{38}{47} \cdot \frac{37}{46}.$$

So, the probability that exactly one card is a heart is

$$1 - \frac{38}{47} \cdot \frac{37}{46} - \frac{9}{47} \cdot \frac{8}{46} = \frac{342}{1081}.$$

It is very typical that there are multiple ways to work out problems like this! This offers you a great chance to check your work.

So what's the probability you win? 0.9 times the probability that exactly one heart comes, plus 0.5 times the probability that two hearts come. In other words,

$$0.9 \times 0.316 + 0.5 \times 0.033 \sim 0.301,$$

which for the sake of simplicity we will round off to 0.3.

Now, on to the expected value computation. If you call and win, then you win \$1,200: the \$500 previously in the pot, plus the \$700 that Alice bet. If you call and lose, you lose \$700. Therefore the expected value of calling is

$$0.3 \cdot 1200 + 0.7 \cdot (-700) = -130.$$

It's negative, so you should fold here.

But notice that it's close! So, for example, if the flop had come $A\heartsuit 8\heartsuit 7\clubsuit$, then you should call. (Exercise: verify this as above!) Here you will make a straight if a six comes. It is not so likely that a six will come, but a small probability is enough to swing your computation.

Example 2. The same situation, except imagine that you both have 1,000 chips remaining and that Alice bets only 300 chips. What should you do?

You could consider folding, calling, or now raising. Let us eliminate raising as a possibilty: if Alice is bluffing with something like $Q\clubsuit 7\diamondsuit$, then you might get her to fold, even though she has a better hand. But this doesn't seem very likely.

Since you have the opportunity to bet again, let us now consider **the next card only.**

- Suppose the next card is a heart, giving you a flush. Then, you think it is more likely than not that you'll win, so you want to bet. Moreover, since Alice might have one heart in her hand, you would really like her to fold – and so if this happens, you will go all in.

  It is difficult to estimate the probabilities of what happens next – this depends on how you see Alice, how she sees you, and what she's holding. As a rough estimate, let us say there is a 50-50 chance that she calls your all-in bet, and if she calls there is a 75% chance of you winning with your flush.

- Suppose the next card is not a heart. Then you don't want to bet, because you don't have anything. Let us say that there is a 75% chance that Alice goes all-in, in which case you should and will fold. (Check the math here!)

If Alice instead checks (assume there is a 25% chance of this), you both get to see one more card and bet again. If it is a heart, assume that you both go all-in and that you win with 75% probability. If it is not a heart, assume that Alice goes all in and you fold.

These percentages are approximate – once again we can't really expect to work exactly. But given the above, we can enumerate all the possibilities, their probabilities, and how much you win or lose:

- Heart, she calls your all-in, you win: probability $\frac{9}{47} \times \frac{1}{2} \times \frac{3}{4} \sim 0.072$, you win \$1500. (The initial \$500 pot, and her \$1000.)
- Heart, she calls your all-in, you lose: probability $\frac{9}{47} \times \frac{1}{2} \times \frac{1}{4} \sim 0.024$, you lose \$1000. (Your remaining \$1000.)
- Heart, she folds: $\frac{9}{47} \times \frac{1}{2} \sim 0.096$, you win \$800. (The initial \$500 pot, plus the \$300 she invested to make the first bet.)
- Not a heart, she goes all-in: $\frac{38}{47} \times \frac{3}{4} \sim 0.606$, you lose \$300. (This is what you invested to call her first bet, but you fold and so avoid losing any more.)
- Not a heart, she checks, next card is a heart, you win: $\frac{38}{47} \times \frac{1}{4} \times \frac{9}{46} \times \frac{3}{4} \sim 0.030$. You win \$1500.
- Not a heart, she checks, next card is a heart, you lose: $\frac{38}{47} \times \frac{1}{4} \times \frac{9}{46} \times \frac{1}{4} \sim 0.010$. You lose \$1000.
- Not a heart, she checks, next card is not a heart, you fold: $\frac{38}{47} \times \frac{1}{4} \times \frac{37}{46} \sim 0.163$. You lose \$300.

As is often the case in poker, it is more likely that you will lose than win, but the winning amounts are larger than the losing amounts. Here there are two reasons for this: first of all, if she goes all-in on a bad card for you, then you can usually fold and cut your losses. The second is that we're comparing against a baseline of folding, which we say has expected value zero. But if you bet, you can not only get Alice to match your bets, but also keep your stake in the existing pot.

The expected value of calling is

$$.072 \times 1500 - .024 \times 1000 + .096 \times 800 - .606 \times 300 + .040 \times 1500 - .013 \times 1000 - .163 \times 300 \sim -35.$$

A close decision, but if we believe our assumptions, then it looks like it's wise to fold.

Example 3. You are the big blind (50 chips) at a full table, playing Texas Hold'em. The first player, who is known to be conservative, raises to 200 chips, and everyone else folds to you. You have a pair of threes, and if you call, both you and your opponent will have 3,000 more chips to bet with. Since you already have 50 chips in the pot, it costs you 150 chips to call.

Should you call or fold?

To solve this problem we again have to make guesses about what we think will happen, which are still more inexact than the last problem. This will set up another expected value problem.

Anyway, the first player is known to be conservative, so she probably has ace-king or a high pair or something like that. Let us assume that *no three comes on the flop, you will not dare to bet.* Assume further that your opponent will, and you end up folding.

Since you have a pair of threes, you are hoping that a three comes on the flop. If so, you will almost certainly win. Let us assume that, if a three comes on the flop:

- With 25% probability, your opponent will fold immediately and you will win the current pot (of 425 chips: your bet, her bet, and 25 chips from the small blind).
- With 60% probability, your opponent will bet somewhat aggressively, but eventually fold, and you win (on average) the current pot of 425 chips, plus an additional 500 chips.
- With 10% probability, your opponent will bet very aggressively. Both of you go all-in, and you win the pot of 425 chips plus all 3,000 of her remaining chips.
- With 5% probability, your opponent gets a better hand than three threes, and both of you go all-in and you lose 3,000 of your remaining chips.

Let α be the probability of a three coming on the flop. Then, the expected value of calling (relative to folding) is

$$-150 + \alpha \cdot \Big(.25 \cdot 425 + .60 \cdot 925 + .10 \cdot 3425 + .05 \cdot (-3000) \Big) = -150 + 853.75\alpha.$$

So we need to compute α to determine whether this is positive or negative. To illustrate our techniques, we will do this in two different ways. In both cases we compute the probability that **no** three comes on the flop, and then subtract this from 1.

Solution 1. The first card will not be a three with probability $\frac{48}{50}$: there are 50 cards remaining, and 48 of them are not threes. If the first card is not a three, then the second card will not be a three with probability $\frac{47}{49}$, and the third card will not be a three with probability $\frac{46}{48}$. The probability that at least one card is a three is therefore

$$1 - \frac{48}{50} \cdot \frac{47}{49} \cdot \frac{46}{48} = .117 \cdots$$

Therefore, the expected value of calling is

$$-150 + 835.75 \cdot .117 = -52.30.$$

It is negative, so a call is more prudent.

Solution 2. We compute in a different way the probability that none of the three cards in the flop is a three. There are $C(50, 3)$ possible flops, and $C(48, 3)$ possible flops which don't contain a three. So this probability is $\frac{C(48,3)}{C(50,3)}$, which is the same as $\frac{48}{50} \cdot \frac{47}{49} \cdot \frac{46}{48}$.

Some remarks:

- If you each had 10,000 remaining chips, then it **would** make sense to call. (Redo the math to see why!!) This illustrates the principle that long-shot bets are more profitable if you possibly stand to make a very large amount of money.
- The above computations assumed that all 50 cards were equally probable. But, given what you know about your opponent, you might assume that she doesn't have a three in her hand. In this case, the probability of getting a three on the flop goes up to

  $$1 - \frac{46}{48} \cdot \frac{45}{47} \cdot \frac{44}{46} = .122 \cdots$$

  which is slightly higher.

5.4 Exercises

Thanks to the participants (credited by their screen names below) in the Two Plus Two Forums for suggesting poker hands which are treated here:

`http://forumserver.twoplustwo.com/32/beginners-questions/videos-online-illustrating-simple-mathematical-poker-concepts-1631031/`

1. Refer to Harrington's opening strategies for Texas Hold'em described above. If you are in middle position and everyone has folded before you, compute the probability that you are dealt a hand which Harrington suggests raising.

   Now do the same for late position.

2. (Suggested by ArtyMcFly.) The following amusing clip shows a hand in a million-dollar Hold'em tournament with eight players, where two players are each dealt a pair of aces. One of them makes a flush and wins.

   `https://www.youtube.com/watch?v=aR52zv1GqBY`

   (a) Compute the probability that Drinan and Katz are each dealt a pair of aces. (No need to approximate; you can compute this exactly.)

   (b) Compute the probability that any two of the eight players are each dealt a pair of aces.

(c) Given that two players are dealt aces, these aces must be of different suits. Each player will win if at least four cards of one of his two suits are dealt. (If four of this suit are dealt, then he will make a flush. If five of this suit are dealt, then both players will have a flush, but only one of them will have an ace-high flush.)

The broadcast lists a probability of 2% of this happening for each player. Compute this probability exactly.

(Note that the most common outcome is that no four cards of the same suit will be dealt, in which case the two players will have equal hands and tie.)

(d) Compute the probability of this whole sequence happening: two of the eight players are dealt a pair of aces, and one of them makes a flush and wins. Please give both an exact answer and a decimal approximation.

(e) Suppose these eight players play one hundred hands of poker. What is the probability that this crazy sequence of events happens at least once?

3. (Suggested by whosnext.) Here is another clip illustrating some serious good luck. (Or bad luck, depending on whose perspective you consider!)

   `https://www.youtube.com/watch?v=72uxvL8xJXQ`

   Danny Nguyen is all-in with $A\diamondsuit 7\diamondsuit$ against an opponent with $A\spadesuit K\clubsuit$. The flop is $5\heartsuit K\heartsuit 5\spadesuit$. After this, the next two cards must both be sevens for Nguyen to win. Compute the probability of this happening.

   (Note: there is also a small possibility of a tie, for example if both cards are aces.)

4. Consider a variant of poker where you are dealt four cards instead of five. So a 'straight' consists of four consecutive cards, a 'flush' four of a suit.

   By analogy with ordinary poker, determine what the possible hands are, and determine the probability of each. For each hand, give an exact answer for the probability as well as a decimal approximation.

5. (This is Hand 4-3 from *Harrington on Hold'em, Volume 1.*)

   Early in a poker tournament, with blinds \$5 and \$10, you are sitting third out of ten players in a no-limit Hold'em tournament with a stack of \$1,000. You are dealt $A\diamondsuit K\spadesuit$.

   The first two players fold, and you elect to raise to \$50. The next four players fold, and the eighth (next) player, who has a stack of \$1,630, calls your bet. The total pot is \$115, and the remaining players fold.

   The flop comes $J\diamondsuit 7\clubsuit 4\heartsuit$, and you act first. You choose to bet \$80. (This is a 'continuation bet', a kind of bluff. Since you expect that your opponent is somewhat likely to fold, this is considered good strategy.)

   Your opponent raises to \$160. **Do you fold, call, or raise the bet?**

You should analyze this hand as in the examples in the book and in lecture. As best as you can, estimate your odds of having the best hand after the turn and the river, and carry out an appropriate expected value computation.

Note: There is no single right answer, so justify your assumptions. If you like, you may work with one other person in the class and turn in a joint soution to this problem.

6. (This is the optional bonus.) Watch part of the World Series of Poker clip in the text, or any other poker tournament which is publicly available. (With your solution, please let me know where I can find video to watch the hand myself.) Find a decision made by one of the players similar to the situation in the text or the previous problem, and either explain or critique the play. Your solution should involve probability and expected value computations somewhere!

6 Inference

Your instructor decides to conduct a simple experiment. He pulls out a coin and is curious to see how many consecutive heads he will flip. He starts flipping – and lo and behold he flips a long sequence of consecutive heads! Six, seven, eight, nine, ten .... What are the odds of consecutive heads? $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$. Pretty unlikely!

He continues flipping. Eleven, twelve, thirteen, fourteen, ... the probabilities get smaller and smaller. But eventually it occurs to you that there is an alternative, and indeed more likely, explanation: *You cannot see the coins*, and so *perhaps your instructor was just lying to you.*

What happened here? After the first coin, or after the second, you probably didn't suspect any dishonesty – after all, it is not so unlikely to flip one or two heads. He *could have been* lying, but you probably didn't suspect that. But while the probability of umpteen consecutive heads goes down and down, the probability that he was lying from the beginning doesn't, and eventually the latter becomes more plausible.

This is an example of *Bayesian inference*, which we will explore from a mathematical point of view. But even if you don't know the mathematics yet, you already make similar inferences all the time. For example, suppose that a politician makes a claim you find surprising.[6] Then, informally you will assess the probability that the claim is true. In doing so, you will take into account two factors: (1) how likely you believed this claim might have

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[6] More specifically, this claim should concern a *matter of fact*, which can be independently verified to be true or false. For example, a politican might claim that crime levels have been rising or falling, that the moon landing was faked, or that Godzilla was recently sighted in eastern Siberia. Even if such claims cannot be confirmed or denied with 100% accuracy, the point is that they are objectively true or false. This is different than offering an opinion or speculation. For example, a politician might claim that if we airlift ten million teddy bears into North Korea, they will overthrow their dictator and become a democracy. We cannot say this is true or false without trying it. Similarly, a politician might say that Americans are the kindest people in the world. Unless you are prepared to objectively measure 'kindness', this is a subjective matter of opinion.

been true, before the politician made it; (2) your assessment of the honesty of the politician in question.

And finally we can look for examples from game shows. Here is a clip of *Let's Make a Deal*:

`https://www.youtube.com/watch?v=-vRty_kkfgw`

What would you do?

6.1 Conditional Probability

Definition 6.1 *Let A and B be events in a sample space S. If $P(A) \neq 0$, then the* **conditional probability of B given** *A, written $P(B|A)$, is*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Here the symbol $\cap$ means **intersection** – so $A \cap B$ is the set of outcomes that are in both A and B. In informal language, $P(A \cap B)$ is the probability that both A and B occur. We also sometimes omit the word 'conditional', and just say 'the probability of B given A'.

Example 6.2 *You flip two coins. Compute the probability that you flip at least two heads, given that you flip at least one head.*

Solution. We could give a quicker solution, but let's write out everything explicitly for clarity's sake. The sample space is

$$S = \{HH, HT, TH, TT\},$$

with all outcomes equally likely. Call A the event that we flip at least one head. We have

$$A = \{HH, HT, TH\}.$$

Write B for the event that we flip two heads. We have

$$B = \{HH\}.$$

We also have

$$A \cap B = B = \{HH\},$$

because B is a subset of A. (*Warning: In many examples B will not be a subset of A.*)

So,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Note that we could have simplified the arithmetic in the last step. We have $P(A \cap B) = \frac{N(A \cap B)}{N(S)}$, and also $P(A) = \frac{N(A)}{N(S)}$, so that

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{N(A \cap B)}{N(S)}}{\frac{N(A)}{N(S)}} = \frac{N(A \cap B)}{N(A)}.$$

This is an alternative formula which we can use whenever it's easier. (It holds always, not just for this example.)

The next example concerns the Price Is Right game **One Away.**

`https://www.youtube.com/watch?v=V6gCNW5wFIY`

Game Description (One Away – The Price Is Right): The contestant is shown a car and a five digit price for the car. Each digit in the price is off by one – too low or too high. She then guesses the price of the car, one digit at a time.

If her guess is correct, she wins the car. Otherwise, if at least one digit is correct, she is told how many digits she has right and can make corrections as she sees fit.

Example 6.3 *Suppose that all the digits are random. Compute the probability that her first guess is correct, given that she has at least one number right.*

Solution. This is easy: Let A be the event that she has at least one number right, so $P(A) = \frac{31}{32}$, an let B be the event that she has all five right, with $P(B) = P(B \cap A) = \frac{1}{32}$. We have

$$P(B|A) = \frac{P(B)}{P(A)} = \frac{1}{31}.$$

Her winning chances have gone from 1 in 32 to 1 in 31, *if* you assume the digits are random. In reality, we can be pretty sure the first digit is indeed a 2. (If this is not clear to you, please take a friend with you next time you go car shopping.) If the *other* digits are random, her winning chances have gone from 1 in 16 to ...... 1 in 16.

She is then told she has at least two numbers right.

Example 6.4 *Suppose that all the digits are random. Compute the probability that her first guess is correct, given that she has at least two numbers right.*

Solution. Now writing A for the event that she has at least two numbers right, we have $P(A) = \frac{26}{32}$ and so $P(B|A) = \frac{1}{26}$.

How did we compute 26? Well, the number of ways to get *exactly* two numbers right is $C(5, 2)$: choose which two numbers. And so, the number of ways to get at least two numbers right is

$$C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5) = 10 + 10 + 5 + 1.$$

Now here's what no contestant on this show realizes: she should be **ecstatic** that she **doesn't** have two numbers right. Let us go back to assuming the first digit is definitely a 2, and the others are random. Her initial winning chances are 1 in 16, and once we know she has at least two numbers right they go up to 1 in 15. Whoopty doo. But, if she *doesn't* have at least two numbers right, then *keep the initial* 2 *and change everything else!!*

Example 6.5 *You roll two dice. What is the probability that the sum of the numbers showing face up is* 8*, given that both of the numbers are even?*

Note that, in contrast to the previous examples, neither event is implied by the other. We could roll an 8 without either of the numbers being even, and we could roll two even numbers whose sum isn't 8.

Solution. Writing S for the sample space, it has 36 elements as we have seen before. Write A for the event that the numbers are both even, and B for the event that the total is eight. Then we have

$$A = \{22, 24, 26, 42, 44, 46, 62, 64, 66\},$$

$$B = \{26, 35, 44, 53, 62\},$$

$$A \cap B = \{26, 44, 62\}.$$

Then (using the alternate version of our formula) we have

$$P(B|A) = \frac{N(A \cap B)}{N(A)} = \frac{3}{9} = \frac{1}{3}.$$

In conclusion, if we know that both numbers are even, this makes it more likely that they will sum to eight. This is true even though we removed some possibilities like $3 + 5$.

6.2 The Monty Hall Problem

We now consider the most famous mathematical problem coming from a game show: the *Monty Hall Problem*. Although the problem was based on *Let's Make a Deal*, the problem was made up and never actually happened on the show. But it's a good problem, so we'll consider it anyway.

The Monty Hall Problem. Monty Hall, on Let's Make a Deal, shows you three doors. Behind one door is a car, behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to switch to door No. 2?"

Is it to your advantage to switch your choice?

Let's assume that the contestant prefers the car to a goat.[7] We will make the following further assumptions:

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[7] But see `https://xkcd.com/1282/`.

- Initially, the car is equally likely to be behind any of the three doors.
- After you choose a door, the host will randomly pick one of the other doors with a goat and open that one.

  More specifically: If you choose a door with a goat, then exactly one of the other two doors will have a goat and the host will show it to you. If you choose the door with the car, then both of the other doors will have goats and the host will pick one of them at random and show it to you.

So, given that you choose Door 1, let's compute the sample space of all possible outcomes:

- The car is behind Door 2 (probability $\frac{1}{3}$). Monty shows you Door 3.
- The car is behind Door 3 (probability $\frac{1}{3}$). Monty shows you Door 2.
- The car is behind Door 1, and Monty shows you Door 2. (Probability $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$).
- The car is behind Door 1, and Monty shows you Door 3. (Probability $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$).

Let B be the event that the car is behind Door 2 (so $P(B) = \frac{1}{3}$), and let A be the event that Monty shows you Door 3. We want to compute $P(B|A)$, the probability that the car is behind Door 2, given that Monty showed you Door 3.

We have

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

The probability $P(A \cap B)$ is $\frac{1}{3}$, the same as $P(B)$. As we saw before, if the car is behind Door 2, Monty will always show you Door 3.

The probability $P(A)$ is $\frac{1}{2}$, the sum of the two probabilities above in which Monty shows you $\frac{1}{2}$. If the car is behind Door 2, Monty will always show you Door 3, and if the car is behind Door 1 then Monty *might* show you Door 3.

So

$$P(B|A) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Given that Monty showed you Door 3, there is now a $\frac{2}{3}$ probability the car is behind Door 2. You should switch.

Many people find this counterintuitive, and so we will consider some alternative fomulations.

Monty Hall with a Million Doors. Instead of three doors, Monty shows you one million, and randomly you choose door 816,280. Monty then opens 999,998 of the remaining doors – all but your door and Door 161,255. He offers you the opportunity to switch.

You really feel like you should take it, don't you?

Bertrand's Box is a closely related paradox. There are three boxes – one with two gold coins, one with two silver coins, and one with one of each. You don't know which box is which. You randomly choose one of the boxes and one of the coins in it, and it turns out that your coin is gold. What is the probability that the other coin in the box is also gold? Answer: Two thirds. (Work it out!)

The Prisoner Paradox was posed by Martin Gardner in 1959, and is equivalent to the Monty Hall problem. Here it is, in Gardner's original formulation.[8]

> A wonderfully confusing little problem involving three prisoners and a warden, even more difficult to state unambiguously, is now making the rounds. Three men-A, B and C-were in separate cells under sentence of death when the governor decided to pardon one of them. He wrote their names on three slips of paper, shook the slips in a hat, drew out one of them and telephoned the warden, requesting that the name of the lucky man be kept secret for several days. Rumor of this reached prisoner A. When the warden made his morning rounds, A tried to persuade the warden to tell him who had been pardoned. The warden refused. 'Then tell me,' said A, 'the name of one of the others who will be executed. If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C.'
>
> Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. 'If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C.'
>
> 'But if you see me flip the coin,' replied the wary warden, 'you'll know that you're the one pardoned. And if you see that I don't flip a coin, you'll know it's either you or the person I don't name.'
>
> 'Then don't tell me now,' said A. 'Tell me tomorrow morning.'
>
> The warden, who knew nothing about probability theory, thought it over that night and decided that if he followed the procedure suggested by A, it would give A no help whatever in estimating his survival chances. So next morning he told A that B was going to be executed.
>
> After the warden left, A smiled to himself at the warden's stupidity. There were now only two equally probable elements in what mathematicians like to call the 'sample space' of the problem. Either C would be pardoned or himself, so by all the laws of conditional probability, his chances of survival had gone up from 1/3 to 1/2.
>
> The warden did not know that A could communicate with C, in an adjacent cell, by tapping in code on a water pipe. This A proceeded to do, explaining to C

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[8] Gardner, Martin (October 1959). *Mathematical Games: Problems involving questions of probability and ambiguity.* Scientific American. 201 (4): 174–182. Available online: `http://www.nature.com/scientificamerican/journal/v201/n4/pdf/scientificamerican1059-174.pdf`.

exactly what he had said to the warden and what the warden had said to him. C was equally overjoyed with the news because he figured, by the same reasoning used by A, that his own survival chances had also risen to 1/2.

Did the two men reason correctly? If not, how should each calculate his chances of being pardoned? An analysis of this bewildering problem will be given next month.

6.3 Bayes' Theorem

Suppose you go caving: you explore all sorts of beautiful underground caverns, and have a fabulous time. But afterwards you are alarmed to hear that some cavers catch the disease of *cavitosis*.[9] The disease can be treated, and so you decide to be tested to see if you have caught the disease.

You learn the following:

- One in a thousand cavers develop cavitosis, and so your *a priori* probability of having cavitosis is $\frac{1}{1000}$.
- The test is not completely reliable, and has a *false positive* rate of 1%. This means that if you don't have the disease, and you get tested for it, then with probability $\frac{99}{100}$ you will be told that you don't have the disease and with probability $\frac{1}{100}$ you will be told (incorrectly) that you do have it.
- The test has a *false negative* rate of 3%. This means that if you *do* have the disease, and you get tested for it, then with probability $\frac{97}{100}$ you will test positive[10] and with probability $\frac{3}{100}$ you will test negative.

There are four probabilities we can compute:

- The probability that we have the disease, and test positive for it, is
$$\frac{1}{1000} \times \frac{97}{100} = \frac{97}{100000} = 0.097\%.$$
The probability that we do have the disease, but test negative for it, is
$$\frac{1}{1000} \times \frac{3}{100} = \frac{3}{100000} = 0.003\%.$$
The probability that we don't have the disease, but test positive for it, is
$$\frac{999}{1000} \times \frac{1}{100} = \frac{999}{100000} = 0.999\%.$$
The probability that we don't have the disease, and test negative for it, is
$$\frac{999}{1000} \times \frac{99}{100} = \frac{98901}{100000} = 98.901\%.$$

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[9] This is completely made up.

[10] 'Positive' does not mean 'good'; it means that you do have whatever condition was being tested for.

Example 6.6 *You get tested for cavitosis, and to your horror the test comes back positive. Compute the probability that you do indeed have cavitosis.*

Solution. The probability that you have cavitosis, and test positive for it, is $\frac{97}{100000}$.

The probability that you test positive for cavitosis is the sum of the two relevant probabilities above:

$$\frac{97}{100000} + \frac{999}{100000} = \frac{1096}{100000}.$$

Therefore, the probability that you have cavitosis, given that you have tested positive for it, is

$$\frac{97}{1096} = 8.85 \cdots \%.$$

In other words, you should be concerned, and if your doctor prescribes antibiotics then you should take them, but *it is still more likely than not that you don't have the disease.* In particular, it is much more likely that the test resulted in a *false positive.*

A false positive rate of 1 in 100 sounds pretty good, but *out of those who test positive* this results in a 11 in 12 failure rate of the test!

We now state a theorem that could have been used to derive this result more quickly.

Theorem 6.7 (Bayes' Theorem) *Suppose that A and B are any two events. Then we have*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

It is easy to understand why this is true. The left hand side is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

and the right side is equal to

$$\frac{\frac{P(B \cap A)}{P(A)} P(A)}{P(B)},$$

which we immediately see is the same thing.

Example 6.8 *Here is an example borrowed from Wikipedia. Suppose that the probability that any one person has cancer is 1 in 100.*

Now suppose a 65 year old goes to the doctor to see if she has cancer. She knows that 0.2% of all people are age 65, and of those who test positive for cancer, 0.5% are age 65.[11]

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[11] Note that the above probabilities are all things one could probably look up in books or on the internet!

Solution. We can apply Bayes's Theorem to compute the probability that she has cancer. Let A be the event that she has cancer, and B be the event that she is 65 years old.[12] Then, we have

$$P(A) = 0.01, \quad P(B) = 0.002, \quad P(B|A) = 0.005,$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.005 \times 0.01}{0.002} = 0.025.$$

So there is a 2.5% chance she (a random 65-year-old) has cancer.

Note that we can rewrite Bayes's rule slightly as

$$P(A|B) = P(A) \times \frac{P(B|A)}{P(B)},$$

which allows us to to interpret Bayes' theorem more natiurally. Here $P(A)$ is the *prior* (or *base*) probability of having cancer, i.e., the probability that a person about whom you don't know anything has cancer. The ratio $\frac{P(B|A)}{P(B)}$, then, tells you how much more (or less) likely knowing B makes A. For example, in the example above we have

$$\frac{P(B|A)}{P(B)} = \frac{0.005}{0.002} = 2.5.$$

So we can say that *being 65 years old makes it 2.5 times as likely that you will have cancer.*

Example 6.9 *We apply this to the cavitosis example above. Let A be the event that we have cavitosis, and let B be the event that we test positive for it. We are interested in computing* $P(A|B)$ *– the probability of having the disease, given a positive test.*

Solution. By Bayes's theorem, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.97 \times \frac{1}{1000}}{???}$$

Here we need to compute $P(B)$. It isn't given to us, and we need to compute it in the same way we did above. There are two ways we could test positive for cavitosis: either we have the disease and tested positive for it, or we don't but got a false positive.

In other words, we have

$$P(B) = P(B|A)P(A) + P(B|\neg A)P(\neg A).$$

Here the symbol $\neg$ means *not*. The above formula enumerates the two ways in which we might have tested positive for cavitosis.

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[12] What is it mean to talk about $P(B)$ here? After all, if we've said she's 65 years old, then the probability of her being 65 is 100%, right?

The proper interpretations of these probabilities are as proportions of the population at large. This is the context in which our 0.2% and 0.5% estimations make sense.

These probabilities we all computed above, and we have

$$P(B) = P(B|A)P(A) + P(B|\neg A)P(\neg A) = 0.97 \times \frac{1}{1000} + 0.01 \times \frac{999}{1000} = .01096.$$

So we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.97 \times \frac{1}{1000}}{.01096} = \frac{97}{1096} = 0.0885\ldots$$

Note that *this is the same way we computed it before*, only introducing notation and the formalism of Bayes' Theorem. The theorem hopefully helps us better understand the *principle* of the computation we did before.

Note that it is interesting to write Bayes's theorem in the form

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}.$$

In the denominator we enumerate, and compute the probabilities of, the two ways in which B can happen, and the numerator shows just the one that involves A being true. (Sometimes we have to enumerate more than two possibilities in the denominator.)

Elections and polling. As this chapter was being written, the U.S. 2016 Presidential election was underway[13]. The candidates are Hillary Clinton and Donald Trump, and the race is receiving an *extraordinary* amount of attention.

A popular site is **FiveThirtyEight**, started by Nate Silver:

`http://fivethirtyeight.com`

Silver (and others writing for the site) have created a mathematical model which uses polling data to track the outcome of the election, and Bayesian inference is at the heart of how Silver's model works.

For example, consider the following (hypothetical) situation. Imagine that Clinton and Trump are tied on the eve of the first debate, and at the first debate Clinton promises to give every American a free puppy. How will Americans react to this? Maybe they think puppies are adorable and now are eager to vote for Clinton. Maybe they dread cleaning up all the dog poop, and are thus more inclined to vote for Trump. If you're Nate Silver, you don't try to figure this out – you just pay attention to the polls.

Polls have a *margin of error*. Suppose you poll 1000 people, among a population which is divided 50-50. What's the probability at least 520 express support for Clinton?[14] Roughly

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[13] and we refer to it henceforth in the present

[14] In a sense, you know how to do this already. It is the same as the probability of flipping a coin a thousand times, and getting 520 or more heads, so

$$\frac{C(1000, 520) + C(1000, 521) + C(1000, 522) + \cdots}{2^{1000}}.$$

But that's a mess to compute, and we're interested in approximate (and, as it turns out, extremely close) models.

speaking, the margin of error is about $\frac{1}{\sqrt{n}}$, where n is the number of people polled. So, if you poll 1000 people, your margin of error is about $\frac{1}{\sqrt{1000}} = 0.0316\cdots = 3.16\%$. The true error could be higher – by some freak accident, you could reach a thousand Trump supporters in a row, just like you *could* flip a coin and get a thousand consecutive tails. The margin of error represents the range of outcomes you wouldn't be too surprised by. So, here in this 1,000 person poll, you expect roughly 470 to 530 to express support for either candidate. If you get 550 Clinton supporters, that is pretty strong evidence that public opinion has shifted. If you get 520 Clinton supporters, then that is *some* evidence that public opinion has shifted.

To give a flavor of how these computations look, consider the following simplified model of polling: we assume that there is a 60% chance that a poll is accurate, a 20% chance that it is three points too high, and a a 20% chance that it is three points too low.

Meanwhile, back to our debate and Clinton's puppy promise.

Example 6.10 *We'll say that there is a 30% chance this lowered Clinton's support three points, 30% chance that it raised it three points, and 40% that it didn't make a difference.*

Suppose then that a post-puppy poll comes out showing Clinton at 53%. What is the probability that her support actually increased?

Solution. Let A_1, A_2, and A_3 be the events that Clinton is now at 53%, 50%, and 47% respectively, and B be the event that she polled at 53%. We want to compute $P(A_1|B)$, and we have

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)}.$$

We know every quantity on the right side of this equation. We get

$$P(A_1|B) = \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.2 \times 0.4 + 0 \times 0.3} = \frac{0.18}{0.26} = \frac{18}{26} = 69.2\ldots\%.$$

Note the zero in the denominator – according to our model, polls can not be off six points so we know Clinton's support didn't increase. The two possibilities are that Clinton's support genuinely went up, or that the polling was too high, and our computation tells us that the first is a somewhat more likely outcome.

In real life the problem is (essentially) *continuous* rather than *discrete*: the poll could have been 51.2%, or 50.4%, or 47.7%, or .... and there is no theoretical limit to how much it can be off. To learn more about how to adjust for this probability, I recommend a course in statistics.

6.4 Monty Hall Revisited

We now return to further discussion of the Monty Hall problem. Our first order of business is to give a solution using Bayes' Theorem.

Again suppose that you have chosen Door 1, and Monty opens Door 3 (which contains a goat) and offers you the opportunity to switch. Should you switch?

Let B be the event that the car is behind Door 2, and let A be the event that Monty shows you Door 3. The probability that switching will pay off is $P(B|A)$ – the probability that the car is behind Door 2, given that Monty showed you Door 3.

We have[15]

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

In this formula, we have:

- $P(A|B)$ is 1. We're assuming here that Monty shows you a door, other than the one you picked initially, which does not have the car. If the car is behind Door 2, then this must be Door 3.
- $P(B)$ is $\frac{1}{3}$ – this is the *initial* probability the car was behind door 2. This is one in three for all the doors.
- $P(A)$ is $\frac{1}{2}$ – if Monty's behavior is random, and we don't know where the car is, then Monty is equally likely to show you either of the two doors you didn't pick.

So we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

This is perhaps more illuminating in the form

$$P(B|A) = P(B) \times \frac{P(A|B)}{P(A)}.$$

Here $P(A)$ is $\frac{1}{2}$. We don't know which of the two remaining doors Monty will show us. But *if* the car is behind Door 2, then this *raises* the probability that he'll show you Door 3. Since this is what you in fact observed, it makes it more likely that the car was behind Door 2.

To help further understand this, now let B be the event that the car is behind Door 1. This is less likely than Door 2, and we can see why our previous computation changes:

- $P(B)$ is still $\frac{1}{3}$ – this is the *initial* probability the car was behind door 1. This is one in three for all the doors.
- $P(A)$ is still $\frac{1}{2}$ – if Monty's behavior is random, and we don't know where the car is, then Monty is equally likely to show you either of the two doors you didn't pick.
- $P(A|B)$ is now $\frac{1}{2}$ instead of 1. Since the car is assumed to be behind Door 1, Monty chooses one of the two remaining doors at random. He is not forced to show you Door 3.

---

[15] Note that the roles of A and B are switched here.

In other words, the *prior* probabilities of being behind Door 1 or Door 2 were each $\frac{1}{3}$. The equation is: did the fact that Monty showed you Door 3 make either of these more likely? It did *not* make Door 1 more likely, but it *did* make Door 2 more likely because this *would have forced Monty to show you Door 3*, i.e., it made the *sequence of events you actually observed more likely.*

Just how does Monty behave? If you think you've *finally* understood the Monty Hall problem, now I'm going to confuse you.

The Monty Hall Problem – Zonk! Monty shows you three doors, behind one of which is a car. You pick Door #1, and Monty shows you Door #3, behind which is – the car!

You lose. Zonk.

This violates the *assumptions* we made about Monty's behavior. But are we so sure they were correct?

Jeffrey Rosenthal introduces[16] two variations of the Monty Hall problem. The first is the

The Monty Fall Problem: In this variant, once you have selected one of the three doors, the host slips on a banana peel and accidentally pushes open another door, which just *happens* not to contain the car. *Now* what are the probabilities that you will win the car if you stick with your original selection, versus if you switch to the remaining door?

The Monty Crawl Problem: As in the original problem, once you have selected one of the three doors, the host then reveals one non-selected door which does not contain the car. However, the host is very tired, and *crawls* from his position (near Door #1) to the door he is to open. In particular, if he has a choice of doors to open (i.e., if your original selection happened to be correct), then he opens the *smallest number* available door. (For example, if you selected Door #1 and the car was indeed behind Door #1, then the host would always open Door #2, never Door #3.) What are the probabilities that you will win the car if you stick versus if you switch?

The Monty Crawl Problem is easy: By assumption, Monty *definitely* would have opened Door #2 if it didn't contain the car. Since he skipped by it, you deduce that it contains the car.

For the Monty Fall Problem, again let B be the event that the car is behind Door 2, and let A be the event that Monty shows you Door 3, *and reveals a goat.* The probability that switching will pay off is $P(B|A)$ – the probability that the car is behind Door 2, given that Monty showed you Door 3.

We have still

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

and now

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[16] Problem statements produced verbatim from his *Monty Hall, Monty Fall, Monty Crawl*, Math Horizons, September 2008. Also available here: `http://probability.ca/jeff/writing/montyfall.pdf`

- $P(B)$ is $\frac{1}{3}$, same as it ever was.
- $P(A|B)$ is $\frac{1}{3}$: Monty trips and opens one of the doors. Door 3 is as likely as the other two, so the probability is $\frac{1}{3}$.
- $P(A)$ is now $\frac{2}{9}$: The probabilities that Monty shows you Door 3, and that it contains a goat, are now independent. Their product is $\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$.

So the conditional probability is

$$P(B|A) = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{2}{9}} = \frac{1}{2}.$$

This should be intuitive. There is nothing deliberate about Monty's decision, nothing to inform you that Door 1 or Door 2 has become more likely. But do not in this case that the probability of *Door 1* containing the car has *gone up* (also to $\frac{1}{2}$). Here, there was some possibility that Monty's slip and fall might have revealed the car. The fact that no car appeared meant that the odds of your *existing* choice being correct went *up*.

And, finally, **The Monty Hell Problem.**[17] Monty doesn't want to give you a car. So he makes up his mind to do the following: if you initially pick a door with a goat, then show you the goat. You lose. Zonk. But if you pick the door with the car, then Monty will do everything he can to get you to switch.

In this case, needless to say, if Monty asks you to switch *then you definitely shouldn't.* So the moral is that *the solution to the problem hinges on your assumptions about his behavior.*

6.5 Exercises

The following problems emphasize Bayes' theorem and what it describes about conditional probability. *Instructions*: For each problem, before you work out the details, guess the answer and write down your guess. Then, after you get the answer, compare this to your guess and briefly describe what your competition tells you.

1. Consider the simplified polling example from before. One poll (after Clinton's promise of puppies for all) showed Clinton with 53% support, and we computed that this reflects Clinton's actual level of support with 69.2% probability, and that the candidates are tied with probability 30.8%.

   Suppose now that a second poll comes in, again showing Clinton with 53% support. Now compute the probability that this is her true level of support.

   Finally, suppose that a third poll comes in, this time showing the candidates tied. Now compute the probability that this is her true level of support.

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[17] Taken from the Wikipedia page. See also the very enligthening article by John Tierney in the *New York Times*, July 21, 1991. Available here: `http://www.nytimes.com/1991/07/21/us/behind-monty-hall-s-doors-puzzle-debate-and-answer.html`

When FiveThirtyEight adjusts each candidate's probability of winning, this is what it is doing!

2. Recall the experiment we conducted in class: I flipped a coin repeatedly, and it came up heads every time! Eventually it occurred to you that the experiment was rigged – I was ignoring the actual result of the coin flip and just telling you that it was heads every time.

   Assume there is a prior probability of 95% that I was conducting the experiment honestly, and a 5% chance that I was cheating and would always say it comes up heads. For each $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, after you have seen n heads, compute the probability that the experiment is rigged.

   (Hint: around $n = 4$ or 5, you should definitely suspect it but be unsure. By $n = 10$, there is a small possibility I am telling the truth, but at this point you should be reasonably confident that I'm lying.)

3. You decide to conduct your own experiment: You pull a coin out of your pocket, keep flipping it, and it keeps coming up heads over and over!

   This time, since you produced the coin yourself, you estimate the prior probability as one in a million that by some freak chance this coin had heads on both sides. Now estimate the probability of this, after $n = 0, 5, 10, 15, 20, 25, 30$ flips.

4. As a Gamecock football fan, in August you look at Clemson's team and you wonder if they are incompetent. You guess that with 50% probability they are and with 50% probability they aren't.

   Clemson's first game of the season is against Podunk State. If they are incompetent, they have a 20% chance of winning; otherwise, they have a 70% chance of winning.

   Suppose that Clemson loses its game. How do you revise your estimates as to the probability that Clemson is incompetent?

5. ~~In a variation of Let's Make A Deal, Monty shows you *four* doors, one of which contains a car. You pick Door 1, and then Monty chooses at random one of the other doors which does not contain a car (let's say Door 4) and shows you that it contains a goat instead.~~

   ~~The prior probabilities of each door containing the car were $\frac{1}{4}$. What are the revised probabilities now, and should you switch?~~

6. Here is a clip of the game Barker's Markers (from The Price Is Right):

   `https://www.youtube.com/watch?v=XlTaZlk3mz8`

   (a) Assume that the contestant has no idea how much the items cost and guesses randomly. Also assume that the producers choose randomly two of his correct guesses and reveal them.

Explain why he has a winning probability of $\frac{3}{4}$ if he switches. (Use Bayes' theorem!)

   (b) Watch several clips of this game (also known as 'Make Your Mark' during the Drew Carey era). Determine, as best as you can, the extent to which these assumptions are accurate. In the clips you watch, do you think the contestant should indeed switch?

7. ~~Consider a further variation of Let's Make a Deal with four doors, this time *two* of which contain cars. You pick Door 1, and this time Monty opens a different door which contains a *car*. (If only one of the remaining doors contained a car, he shows you that one; if two did, he chooses one of them at random.)~~

   ~~Compute now the probabilities that each of the doors contains a car. Should you switch?~~

8. Compose, and solve, your own problem that involves Bayes's theorem, similar to the above. (Bonus points for problems which are particularly interesting, especially realistic, or are drawn from any game show.)

9. **Term project.** Send me, **by e-mail**, a rough indication of what you would like to do for your term project. You are free to ask me questions, or to suggest more than one idea if you would like advice on which is the most feasible.

   Unless you prefer to work alone, please let me know whom you plan to work with (if you know). If you don't know, I'm happy to match people who have indicated similar interests.

6.6 More Exercises

1. The election continues, and in a following debate Trump counters Clinton's promises by promising a free kitten to every American family. You initially estimate that with 40% probability Trump now has a 50% level of support, and with 30% probability Trump now has a 52% or 48% level of support.

   In this problem we consider a different polling model. With 60% probability a poll will be accurate; with 15% probability it will be too high by 2 points; with 5% probability it will be too high by 4 points; with 15% probability it will be too low by 2 points; with 5% probability it will be too low by 4 points.

   Four polls come in and show the following levels of respective support for Trump: 50%, 52%, 48%, 54%. After each poll, compute the probability that Trump's support is respectively at 48%, 50%, or 52%.

2. You sit down for a game of poker with a player you have never met before. Suppose that, when first to act, an *aggressive* player will raise 70% of the time and fold 30% of the time; a *conservative* player will raise 40% of the time and fold 60% of the time.

You estimate initially that there is a 50% chance that your opponent is aggressive. The first three times that she is first to act, she elects to raise.

After each raise, compute the revised probability that your opponent is aggressive.

3. You are a child worried about monsters under your bed. Suppose that your knowledge of Bayesian probability outstrips your common sense, so you decide to use Bayes' theorem to assauge your worries.

   You initally estimate that with 30% probability there are monsters under your bed, and with 70% probability there aren't.

   Periodically, you ask your father to look under your bed. If there are monsters under your bed, then with 90% probability they will hide and your father won't see anything. With 10% probability your father will indeed see the monsters (and presumably be eaten by them).

   (a) Suppose you ask your father to look, and on three consecutive nights he doesn't see any monsters. What do you now estimate as the probability that there are monsters under your bed?

   (b) How many times must you ask your father to look before you get this probability under 1%?

4. You speculate that your math professor may be a zombie. At the beginning of the class, you estimate that with 20% probability he is a zombie.

   If he is not a zombie, then each class he will give a normal lecture. If he is a zombie, then with 80% probability he will give his lecture as normal, and with 20% probability he will instead devour your brains.

   Twenty lectures pass, and your professor has still not consumed your brains. With what probability do you now estimate that your professor is a zombie?

5. In a variation of Let's Make A Deal, Monty shows you *four* doors, one of which contains a car. You pick Door 1, and then Monty chooses at random one of the other doors which does not contain a car (let's say Door 4) and shows you that it contains a goat instead.

   The prior probabilities of each door containing the car were $\frac{1}{4}$. What are the revised probabilities now, and should you switch?

6. Here is a clip of the game Barker's Markers (from The Price Is Right):

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   (a) Assume that the contestant has no idea how much the items cost and guesses randomly. Also assume that the producers choose randomly two of his correct guesses and reveal them.

Explain why he has a winning probability of $\frac{3}{4}$ if he switches. (Use Bayes' theorem!)

(b) Watch several clips of this game (also known as 'Make Your Mark' during the Drew Carey era). Determine, as best as you can, the extent to which these assumptions are accurate. In the clips you watch, do you think the contestant should indeed switch?

7. Consider a further variation of Let's Make a Deal with four doors, this time *two* of which contain cars. You pick Door 1, and this time Monty opens a different door which contains a *car*. (If only one of the remaining doors contained a car, he shows you that one; if two did, he chooses one of them at random.)

   Compute now the probabilities that each of the doors contains a car. Should you switch?

   Solution. Suppose you pick Door 1 and Monty shows you a car behind Door 2. Let A be the event that a car is behind Door 1. We know $P(A) = \frac{1}{2}$. Let B be the event that Monty shows you a car behind Door 2. We want to compute $P(A|B)$.

   For this we use Bayes' Theorem. We know that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

   We saw earlier that $P(A) = \frac{1}{2}$. What is $P(B|A)$? If a car is behind Door 1, then there is only one car remaining, and Monty is obliged to show it to you. The probability that it is behind Door 2 is $P(B|A) = \frac{1}{3}$.

   Finally, what is $P(B)$? There are two ways that Monty could show you Door 2. The first is that a car is behind Door 1. We just saw that the (prior) probability that a car is behind Door 1 and then Monty shows you Door 2 is $P(B|A)P(A) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

   Alternatively, there is a (prior) probability of $\frac{1}{2}$ that there is no car behind Door 1, in which case the probability Monty shows you Door 2 is still $\frac{1}{2}$. So the total probability of this sequence of this events is $\frac{1}{6}$. The event B takes place if one of these two scenarios happens, so that $P(B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

   We conclude that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3}} = \frac{1}{2}.$$

   Indeed, we knew it had to be $\frac{1}{2}$: you knew that Monty had to show you one of the other three doors, and Door 2 is as good as either of the other two. So $P(A) = P(A|B) = \frac{1}{2}$.

   Now, let A be the event that a car is behind Door 3, with $P(A) = \frac{1}{2}$. As before, let B be the event that Monty shows you a car behind Door 2. We want to compute $P(A|B)$.

   What changes? We compute $P(B|A)$. If a car is behind Door 3, then the second car is equally likely to be behind Door 1, 2, or 4. Monty will only show you a car behind

Door 2 if one is there, *and even then* there is only a one in two chance he will show you this door (since he might show Door 3 instead). So $P(B|A) = \frac{1}{6}$.

Now what is $P(B)$? It is cumbersome to do this the long way, so here is a shortcut solution: you know that Monty will show you one of Door 2, 3, or 4, so without any information (like A) he is equally likely to show you one of these. So $P(B) = \frac{1}{3}$.

In conclusion, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{1}{3}} = \frac{1}{4}.$$

The probability that Door 3 contains a car has gone down to $\frac{1}{4}$*, so you should not switch to it.* By exactly the same reasoning, the probability that Door 4 also contains a car has also gone down to $\frac{1}{4}$.

8. Compose, and solve, your own problem that involves Bayes's theorem, similar to the above. (Bonus points for problems which are particularly interesting, especially realistic, or are drawn from any game show.)

7 Competition

7.1 Introduction

The following clip is from the final round of the British Game Show *Golden Balls*.

`https://www.youtube.com/watch?v=tYYSu6PkyDs`

Game Description (Golden Balls (Final Round)): Two players play for a fixed jackpot, the amount of which was determined in earlier rounds. They each have two balls, labeled 'split' and 'steal'. They are given some time to discuss their strategies with each other. Then, they each secretly choose one of the balls and their choices are revealed to each other.

If both choose the 'split' ball, they split the jackpot. If one chooses the 'split' ball, and the other 'steal', the player choosing 'steal' gets the entire jackpot. If both players choose 'steal', they walk away with nothing.

In the video, the players are competing for a prize of 8,200 pounds (roughly \$11,000 in US currency). We can summarize the game in a two-by-two grid that describes the possible choices for you and for your opponent, and the outcome of each choice.

		You	
		Share	Steal
Opponent	Share	5500	11000
	Steal	0	0

Now, assuming that you don't care about your opponent one way or another, and want only to maximize your own expected payoff, your optimal strategy is clear. If your opponent steals, it doesn't matter whether you steal or share. If your opponent splits, then you do better if you steal. *Therefore, it is clear that you should always choose the steal ball.*

Of course, your opponent will reason in exactly the same way. Therefore, your opponent will deduce that she should choose the steal ball, *and therefore with optimal strategy both of you will choose 'steal' and win nothing.*

Here is an amusing video of the same game being played by two other contestants:

`https://www.youtube.com/watch?v=S0qjK3TWZE8`

You can also find other videos of this game on YouTube, many of which will do somewhat less to affirm your faith in humanity.

Setup and notation. This is an example of a two-player *strategic game*, of the type studied in the subject known as *game theory*. Although this entire course is about games, mathematical 'game theory' refers to this sort of analysis: you have a game involving two or more players, and you have distilled the analysis down to the point where you know what will happen depending on your, and your opponents', choice of strategy.

A *strategic game* consists of the following inputs:

- Two or more players. Here we will only look at two player games.
- For each player, two or more choices of strategy. (The players choose their strategies independently and simultaneously.) For example, in the Golden Balls example, each can choose 'Share' or 'Steal'. Typically, the strategies are the same for each player, but this doesn't have to be the case.
- A *payoff matrix*, such as the one above, describing the payoff to each player.

In the above, we listed only the payoff to the *first* player. To be more precise, we list the payoffs to both players as follows:

		You	
		Share	Steal
Opponent	Share	5500, 5500	11000, 0
	Steal	0, 11000	0

This is an example of a *non-zero-sum game*. Many games (and especially those that we describe colloquially as 'games') are *zero-sum* in the sense that what is good for one player is equally bad for the other player. For example, if one team wins the World Series, then the other team loses. In most board games, there is one winner and everyone else loses.[18]

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[18] This is not true of all board games. For example check out *Republic of Rome*, described here – `https://boardgamegeek.com/boardgame/1513/republic-rome` – where there can be at most one winner, but where it is possible for all players to lose.

7.2 Examples of Strategic Games

We now give some common examples of strategic games and discuss their optimal strategy.

Example 1. The Prisoner's Dilemma. This is essentially the same as the 'Golden Balls' game discussed above, and is perhaps the most familiar example of a strategic game. One formulation[19] of the Prisoner's Dilemma is as follows.

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to: betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

- If A and B each betray the other, each of them serves 2 years in prison.
- If A betrays B but B remains silent, A will be set free and B will serve 3 years in prison (and vice versa).
- If A and B both remain silent, both of them will only serve 1 year in prison (on the lesser charge).

The payoff matrix for this game as follows:

		A	
		Betray	Silent
B	Betray	-2, -2	-3, 0
	Silent	0, -3	-1, -1

Mathematically speaking, the game is trivial. Each prisoner should betray the other. This invites a very serious paradox, as well as questions about whether the model is realistic. (If you betray your accomplice and he remains silent, you had better get the hell out of town as soon as you are released from jail.) It is also interesting if the game is played multiple times consecutively. But we won't pursue these questions here.

Example 2. Rock, Paper, Scissors. This is a familiar game and we can describe its payoff matrix immediately.

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[19] Taken from the Wikipedia article.

		A		
		Rock	Paper	Scissors
	Rock	0, 0	1, -1	-1, 1
B	Paper	-1, 1	0, 0	1, -1
	Scissors	1, -1	-1, 1	0, 0

This is a stupid game of pure luck, but don't tell these people:

`https://www.youtube.com/watch?v=nGYqSqf0yCY`

Note that this is also an example of a *zero-sum game*: a win for you is equivalent to a loss for your opponent.

If you try to get too clever, then your opponent can outwit you. Best to assume your opponent is the smartest person on earth. Therefore, just play a random strategy: play rock, paper, scissors randomly with probability $\frac{1}{3}$ chance each. Then, on average, you can't win, but you can't lose either.

Example 3: Chicken. The game of **Chicken** is illustrated by this clip[20] from the movie *Rebel Without a Cause*.

`https://www.youtube.com/watch?v=u7hZ9jKrwvo`

Two teenagers challenge each other, in front of all of their friends, to the following contest. They start far apart and drive their cars at maximum speed towards each other. If one swerves and the other one does not, then the driver who swerves is the 'chicken' and loses face among his friends, while the other enjoys increased social status.

If neither swerves, the resulting accident kills them both.

The payoff matrix for this game might be described as follows:

		A	
		Swerve	Straight
B	Swerve	0, 0	1, -1
	Straight	-1, 1	-100, -100

Example 4: Final Jeopardy (Simplified).

Here is a clip of the final round from the game show **Jeopardy**:

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[20] Thanks to Kevin Bartkovich, who taught me this subject and who used this very clip for this example.

`https://www.youtube.com/watch?v=p-jFBEozxWk`

And here is a particularly dramatic one:

`https://www.youtube.com/watch?v=8MwVgf2AzcQ`

Game Description (Jeopardy – Final Jeopardy): Three players come into the final round with various amounts of money. They are shown a category and write down a dollar amount (anything up to their total) that they wish to wager.

After they record their wagers, they are asked a trivia question. They gain or lose the amount of their wager, depending on whether their answer was correct. Only the top finisher gets to keep their money.

This is a complicated game to analyze in full detail, so we consider the following toy model of it:

Two players, A and B, start with 3 and 2 coins respectively. Each chooses to wager none, some, or all of her coins, and then flips a coin. If the coin comes up heads, she gains her wager; tails, she loses it.

How should each player play to maximize her chances of winning (counting a tie as half a win)?

We can compute the payoff matrix for the game as follows:

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{A} \\ & & 3 & 2 & 1 & 0 \\ \hline & 2 & \frac{5}{8} & \frac{6}{8} & \frac{5}{8} & \frac{4}{8} \\ \hline B & 1 & \frac{4}{8} & \frac{5}{8} & \frac{6}{8} & \frac{6}{8} \\ \hline & 0 & \frac{4}{8} & \frac{4}{8} & \frac{6}{8} & 1 \\ \hline \end{array}$$

Here only the payoffs for A are listed; the payoffs for B are the negatives of these. (Since we stipulated that each player wants only to win, the game is a zero-sum game: your opponent's gain is your loss.)

This takes a little bit of work to compute. For each combination of wagers there are four possibilities: A and B both gain, A and B both lose, only A gains, only B gains. We add $\frac{1}{4}$ for each such scenario in which A wins, and $\frac{1}{8}$ for each such scenario in which they are tied.

We can see immediately that A should never wager all three coins: she always does at least as well, and depending on B's strategy possibly better, by instead wagering two coins.

But between the other strategies it is not obvious what the players should choose: your best strategy depends on your opponent's best strategy, in a somewhat complicated way. We will come back to this later.

7.3 Nash Equilibrum

These games are very interesting if done *repeatedly* or there are additional factors (such as altruism) involved. We will assume that the game is played *once* and each player plays exclusively for his or her own self-interest.

The obvious question for a game is: **what is the optimal strategy**? We cannot hope to define that precisely, so we define a mathematically rigorous, more precise notion that captures some of what we mean by 'optimal'. We will imagine that our opponent is a genius, or perhaps a mindreader, and focus on playing defense: making sure that our opponent can't outplay us.

We will allow each player to choose a **mixed** (i.e., random) **strategy**. For example, if you are playing Spilt or Steal you might choose the non-random strategy of always playing Steal. But if you are playing Rock, Paper, Scissors and you always make the same play then you can be exploited by a sufficiently clever opponent. If we are playing defensively, we should choose rock, paper, and scissors each with $\frac{1}{3}$ probability.

Definition 7.1 *By a* **mixed strategy** *we mean an assignment of a probability (between* 0 *and* 1*, inclusive) to each possible strategy.*

Definition 7.2 *Suppose you and your opponent each choose a (mixed) strategy for a game. Then these strategies form a* **Nash equilibrum**[21] *if: your current strategy is optimal against her current strategy, and her current strategy is optimal against your current strategy.*

In other words, given your strategy, your opponent can't do any better than her current strategy, and vice versa. (Being at a Nash equilibrium doesn't mean that you couldn't both do better by both adjusting your strategy.)

Example. Suppose you play Split or Steal. Then a Nash equilibrium is both players choosing to steal with probability 1. If your opponent always steals, then it doesn't matter what you do, so actually *every* strategy is optimal for you. You can't do any worse (or better) than always stealing.

Example. Suppose you play Rock, Paper, Scissors. Then a Nash equilibrium is a mixed strategy: both players choosing rock, paper, and scissors with probability $\frac{1}{3}$ each. If you do this, then your opponent's strategy doesn't matter: the game is, on average, a draw.

Note that no other choices of strategy form a Nash equilibrium. Why is this? Suppose that you choose a different strategy. Then the probabilities will be uneven, so let's say your strategy is biased towards Rock. Then your opponent should switch immediately to playing Paper all the time. But then you should switch to playing Scissors all the time! And so your opponent should switch to Rock all the time. And so on, ad infinitum. The game is not at equilibrium.

Example. Here is the game of *Stag Hunt.* You and another player have to choose to either cooperate and hunt a stag, or to hunt a rabbit on your own. You can catch a rabbit

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[21] Named after John Forbes Nash, as depicted in the movie *A Beautiful Mind.*

by yourself, but you need the other player's cooperation to successfully hunt the stag. The payoff matrix is as follows.

		You	
		Stag	Rabbit
They	Stag	2, 2	1, 0
	Rabbit	0, 1	1, 1

Obviously, you both should cooperate to hunt the stag. This choice is a Nash equilibrium: if your opponent has decided to cooperate, then you want to cooperate with her too.

However, both players hunting the rabbit is also a Nash equilibrium. If your opponent has decided to hunt the rabbit instead of the stag, then that is important, but you have to do the same thing (or go without food entirely).

Example. You and another player place a coin heads or tails, each invisibly to the other. You have no chance to communicate beforehand. If you both make the same choice, then you get a reward; otherwise, nothing happens.

The payoff matrix for this game is as follows.

		You	
		Heads	Tails
They	Heads	1, 1	0, 0
	Tails	0, 0	1, 1

The game has two Nash equilibria: you both pick heads, or you both pick tails. This is somewhat similar to the Stag Hunt, but here it is not obvious which you should pick.

Now we come back to Chicken. We describe a less morbid game along the same lines – sort of like Share or Steal, but with a bit more bite. A referee asks you to choose (independently) whether to be nice or mean. If you are both nice, the referee gives you each a dollar. If the other player is nice and you are mean, then the referee gives you her dollar too. But if you are both mean, the referee takes five dollars from both of you.

All of this is described by the following payoff matrix.

		A	
		Nice	Mean
B	Nice	1, 1	2, 0
	Mean	0, 2	-5, -5

Note that ‘both nice’ is not a Nash equilibrium. If the other player is always nice, you should be mean to take her money. Similarly, ‘both mean’ is also not a Nash equilibrium. If your opponent is determined to always be mean no matter what, then you should give in and be nice, so that you avoid losing anything.

However, it *is* a Nash equilibrium for you to always be nice and your opponent to always be mean, or vice versa. If your opponent is always mean, then you stave off the damage by always being nice. Conversely, if you decide to always play nice, then – in the dog-eat-dog world of theoretical game theory, there is no reason for your opponent not to exploit you.

Are there any Nash equilibria in the middle – where you each play a mixed strategy? Suppose your opponent plays a mixed strategy where she is nice with probability α, and mean with probability $1 - \alpha$.

Then, the expected value of you being nice is

$$1 \cdot \alpha + 0 \cdot (1 - \alpha) = \alpha,$$

and the expected value of you being mean is

$$2 \cdot \alpha + (-5) \cdot (1 - \alpha) = 7\alpha - 5.$$

If $\alpha > 7\alpha - 5$, then you should always be nice; if $\alpha < 7\alpha - 5$, then you should always be mean.

The interesting case is when these are equal, i.e., when $\alpha = 7\alpha - 5$. (Doing the algebra, this is the same as saying $6\alpha = 5$, or $\alpha = \frac{5}{6}$.) If your opponent is nice with probability $\frac{5}{6}$, then every strategy has the same payoff: the extra dollar you get from being mean exactly balances the occasional big loss. So you may as well match her strategy, and this is also a Nash equilibrium.

Indeed, in some sense it is the best one. With this mixed strategy, the probabilities of each combination are

		A	
		Nice	Mean
B	Nice	$\frac{25}{36}$	$\frac{5}{36}$
	Mean	$\frac{5}{36}$	$\frac{1}{36}$

and the expected payoff of the game is

$$\frac{25}{36} \cdot 1 + \frac{5}{36} \cdot 2 + \frac{5}{36} \cdot 0 + \frac{1}{36} \cdot (-5) = \frac{30}{36} = \frac{5}{6}$$

to each player. It is not as good as if you both play ‘Always Nice’, but with this choice of strategy you do pretty well, *and* you ensure that you cannot be exploited.

Final Jeopardy. We finally return to our Final Jeopardy model: A has 3 coins, and B starts with two. Each places a wager and either wins or loses that many coins with 50-50 probability, and wants to finish with more coins than their opponent. (A tie counts as half a win.)

$$\begin{array}{cc|c|c|c|}
 & & & A & \\
 & & 2 & 1 & 0 \\ \cline{3-5}
 & 2 & \frac{6}{8} & \frac{5}{8} & \frac{4}{8} \\ \cline{3-5}
B & 1 & \frac{5}{8} & \frac{6}{8} & \frac{6}{8} \\ \cline{3-5}
 & 0 & \frac{4}{8} & \frac{6}{8} & 1 \\ \cline{3-5}
\end{array}$$

Recall that we didn't include the possibility that A wager all three coins because it was *dominated* by the choice of wagering two coins. Wagering two is at least as good in all situations, and in some situations better.

For simplicity we multiply all the payoffs by 8 and subtract 4. This results in a nicer-looking payoff matrix, without changing the essence of the problem.

$$\begin{array}{cc|c|c|c|}
 & & & A & \\
 & & 2 & 1 & 0 \\ \cline{3-5}
 & 2 & 2 & 1 & 0 \\ \cline{3-5}
B & 1 & 1 & 2 & 2 \\ \cline{3-5}
 & 0 & 0 & 2 & 4 \\ \cline{3-5}
\end{array}$$

We want to find a Nash equilibrium. We can simplify this problem by realizing that *B should never wager* 1. Why? The payoffs are respectively 1, 2, 2 *to A* depending on A's strategy. If she is considering doing this, she should flip a coin and wager either 0 or 2 with a 50-50 chance each. This results in an average payoff of 1, 1.5, and 2 which is better for B.

So we can further simplify the matrix:

$$\begin{array}{cc|c|c|c|}
 & & & A & \\
 & & 2 & 1 & 0 \\ \cline{3-5}
 & 2 & 2 & 1 & 0 \\ \cline{3-5}
B & 0 & 0 & 2 & 4 \\ \cline{3-5}
\end{array}$$

We can go one step further. For player A, it is not harmful to wager 1, but she can just as well wager 0 or 2 with 50-50 probability. The average payoffs are identical. So we can eliminate this strategy from A's choices as well to obtain:

		A	
		2	0
B	2	2	0
	0	0	4

Finally, we can find the Nash equilibrium! We want a mixed strategy. So suppose B plays 2 with probability p and 0 with probability $1-p$. Then the payoff to A from playing 2 is

$$2p + 0(1-p) = 2p,$$

and the payoff to A from playing 0 is

$$0p + 4(1-p) = 4(1-p).$$

In a mixed strategy Nash equilibrium, A will be indifferent to these two strategies, so we must have $2p = 4 - 4p$ – solving this yields $p = 2/3$. So B should play 2 with probability $\frac{2}{3}$ and 0 with probability $\frac{1}{3}$.

Conversely, suppose A plays 2 with probability q and 0 with probability $1-q$. Then the payoff to A (i.e. the negative payoff to B) from playing 2 is

$$2q + 0(1-q) = 2q,$$

and the payoff to A from playing 0 is

$$0q + 4(1-q) = 4(1-q).$$

So, similarly, A should play 2 with probability $\frac{2}{3}$ and 0 with probability $\frac{1}{3}$.

7.4 Exercises

1. Consider our original game of Chicken:

		A	
		Swerve	Straight
B	Swerve	0, 0	1, -1
	Straight	-1, 1	-100, -100

Compute the Nash equilibrium for this game (it will be a mixed strategy for both players) and the expected payoff.

2. You and a friend were planning to meet tonight, but your cell phones are both dead and you cannot communicate. You were either going to meet at the park and go jogging, or go to the movie theater and watch a movie. Since you can't communicate, each of you decides to go to either the park or the theater and hope that the other decides the same.

   You would slightly prefer jogging to the movies, and your friend would slightly prefer a movie to jogging. But the top priority for each of you is meeting up, and you would rather do the other's preferred activity together than your preferred activity by yourself.

   Describe this scenario as a strategic game and come up with a suitable payoff matrix. Compute at least one Nash equilibrium corresponding to your payoff matrix.

3. Consider a game like rock-paper-scissors, but only with rock and paper. If you both show the same thing, then you win a prize, and if you show different things, then your opponent wins the same prize. In addition, if you both show rock, then you get an additional bonus prize. (This does not come at your opponent's expense, so although this outcome is better for you than if you both show paper, it is not worse for your opponent.)

   Describe this scenario as a strategic game and come up with a suitable payoff matrix. Compute at least one Nash equilibrium corresponding to your payoff matrix.

   Solution. One possible payoff matrix is the following. (This posits that the bonus prize is worth half the main prize. Similar, but different, solutions are also possible.)

		A	
		Rock	Paper
B	Rock	3, 0	0, 2
	Paper	0, 2	2, 0

   The pure strategies are not Nash equilibria: there is no strategy that both players could agree to, such that neither would want to deviate.

   So we look for a mixed strategy Nash equilibrium. Suppose first that A chooses to play rock with probability α and paper with probability $1 - \alpha$. Then the payoff to B from playing rock is

$$0 \cdot \alpha + 2 \cdot (1 - \alpha) = 2 - 2\alpha,$$

   and the payoff to B from playing paper is

$$2 \cdot \alpha + 0 \cdot (1 - \alpha) = 2\alpha.$$

A should choose her strategy to leave B indifferent between these two options: Set $2 - 2\alpha = 2\alpha$, so $2 = 4\alpha$, so $\alpha = \frac{1}{2}$.

The game is not symmetric, so we analyze the game the other way. Suppose first that B chooses to play rock with probability β and paper with probability $1 - \beta$. Then the payoff to A from playing rock is

$$3 \cdot \beta + 0 \cdot (1 - \beta) = 3\beta,$$

and the payoff to A from playing paper is

$$0 \cdot \beta + 2 \cdot (1 - \beta) = 2 - 2\beta.$$

B should choose her strategy to leave A indifferent between these two options: Set $3\beta = 2 - 2\beta$, so $2 = 5\beta$, so $\beta = \frac{2}{5}$.

In conclusion, A should choose rock half the time, and B should choose rock $\frac{2}{5}$ of the time. Although this was not asked in the question, we compute the expected payoff of the game. To A it is

$$\frac{1}{2} \cdot \frac{2}{5} \cdot 3 + \frac{1}{2} \cdot \frac{3}{5} \cdot 2 = \frac{6}{5},$$

and to B it is

$$\frac{1}{2} \cdot \frac{2}{5} \cdot 2 + \frac{1}{2} \cdot \frac{3}{5} \cdot 2 = 1.$$

8 Backwards Induction

Example. A Money Division Game. Consider the following game. You and another player play for a pot of \$100. You go first, and you can propose any division of the money between the two of you. The other player may then either accept your division, or flip a coin. If she elects to flip and flips heads, then she gets the entire pot; if she flips tails, then neither of you gets anything.

The key to analyzing this game is *backwards induction*: figure out what your counterpart's optimal strategy is, and then base your strategy on that. The expected value (to her) of a coin flip is \$50.00, and therefore you should offer more than that to ensure that she will take your offer. For example, if you must divide the pot into integer amounts, you should offer to give her \$51.00 and keep \$49.00. This is better than a coin flip, so – if she is playing rationally – she will accept your offer.

Now consider a three player version of the same. The three of you play for a pot of \$100. You propose any division of the money between the three of you. The second player either accepts it, or proposes an alternative division of the money. In the latter case, the third player either accepts that or flips a coin.

By what we have just determined, if the second player proposes a division, she should propose to keep \$49.00 and give \$51.00 to the third player. Obviously you want to avoid this outcome since you will get nothing, So you should make a proposal with more than $49.00 for the second player. The best option is to propose $50.00 for yourself and $50.00 for the second player. The second player should then accept this deal.

Another Prisoner Example. You and one other person share a prison cell. You are both very intelligent, and you have exactly the same motives.

One day the jailer comes and paints a mark on each of your foreheads – either red or blue. You don't have any idea what color your mark is, but you can see your cellmate's – it is red. He can also see yours.

The jailor informs you both that either of you may guess the color of your mark. If you guess right, you will be set free, but if you guess wrong, you will be executed. You would very much like to be set free, but you even more don't want to be executed, so neither of you is willing to guess unless you are certain.

Finally, the jailor then tells you: 'At least one of you has a red mark'. After a few moments, you raise your hand and inform the jailor – correctly – that your forehead has a red mark, and you are set free. *How did you know?*

The solution is to consider the problem from your cellmate's perspective. Suppose instead that you had a blue mark. Then your cellmate would see your blue mark. Since he knew that at least one of you had a red mark, he could deduce that it must be him. So he would have immediately guessed that his own mark was red.

He did not do so; therefore your mark is not blue. So you can guess with confidence that it is red.

8.1 The Big Wheel

Part of *The Price Is Right* consists of spinning the famous **big wheel**. It is played twice each show. Most of the stand-alone clips on Youtube feature something unusual happening, so we refer to 13:00 or 30:00 of the following clip.

`https://www.youtube.com/watch?v=qOWfz7ZN6PE`

Game Description (The Big Wheel – Price Is Right): The **Big Wheel** consists of twenty numbers – 5 through 100 (i.e. five cents through a dollar), in increments of five. Three players compete, and the player who spins the closest to a dollar without going over advances to the Showcase Showdown.

The players spin in order. Each player spins once, and then either keeps the result or elects to spin a second time and add the two results. If the result is higher than $1.00, the player is eliminated immediately. The winner is the player who spins the highest without going over. (If two or more players tie, they advance to a tiebreaker.)

In addition, players earn a bonus if they spin exactly a dollar – but we will ignore this.

The natural question is: how should each of the players play? To be more specific, when they are faced with the decision to spin again or not, should they spin again?

In some ways this is like the strategic games dicussed in the previous chapter. But it has one very important difference: *the players play sequentially rather than simultaneously.*

We will assume that *all players understand their best possible strategy and will play it.* With that in mind, we solve this puzzle by *backwards induction*: we start with the last player (who has by far the easiest decision) and work backwards.

8.1.1 Player 3

This is very easy, and on the show you will even observe that Barker and Carey don't ask the contestants what they want to do.

If you spin more than the two previous contestants (or if they busted by spinning more than \$1.00), then you win and obviously you should not spin again. Conversely, if you spin less than one of the two previous contestants, then you lose if you don't spin again, and you might win if you spin again, so obviously you should spin again.

You could tie. If you and one other player are tied with more than 50 cents, then you have 50-50 odds of winning a tie breaker and less than that of not busting, so you should accept the tie and proceed to the tie-breaker. Conversely, if you and one other player are tied with less than 50 cents, you should spin again. (If you are at exactly 50 cents, then it is a tossup.)

Similarly, if you are tied with both other players, you should spin again if the tie is 65 cents or less, and accept the tie at 70 cents or greater.

8.1.2 Player 2 – Example

This is much more subtle, but once we are done we will understand how to work out Player 1's strategy. We begin with a specific example. Assume that the first player spun 60 cents. Obviously if you spin less than 60 on your first spin, you must go again. Suppose you spin 65 cents on your first spin. Should you spin again, or hold? To answer this question we will compute the winning probability in either outcome.

Suppose first that you hold at 65 cents. Then the third player will spin again. If she spins 65 cents or greater, she will hold, where if she spins 60 cents or less she will spin again.

The probability that the third player will win without a tiebreaker is

$$\frac{7}{20} + \frac{12}{20} \cdot \frac{7}{20} = \frac{14}{25} = 0.56.$$

The first figure is the probability that she will win on the first spin; the second is the probability she will take a second spin times the probability she will win on that spin. (Note that, no matter what she spins on the first spin, if it is less than 65 cents there are exactly seven outcomes with which she will win on the second spin, and so seven numbers that will put her between 70 cents and one dollar. Other numbers will leave her too high or too low.)

The probability that the third player will force a tiebreaker is

$$\frac{1}{20} + \frac{12}{20} \cdot \frac{1}{20} = \frac{2}{25} = 0.08.$$

So her winning chances are $0.56 + \frac{1}{2} \cdot 0.08 = 0.6$ – so yours are 0.4, or 40 percent.

Now we consider the option of spinning again. Here we can take a shortcut, and notice that in this case your odds are less than 35%: there is a 35% probability that you will not bust, and if you don't bust then your opponent still has some odds to beat you. So, although we could compute this probability we don't need to.

So, in conclusion, if the first player spins 60 cents on her first spin, and you spin 65 cents on your second spin, you should keep it and not attempt to spin again. Clearly this is still more true if you spin more than 65 cents on your spin, and if you spin less than 60 cents then you should go again.

What if you tie the first player? If you hold, the probability of a third player victory without a tiebreaker is

$$\frac{8}{20} + \frac{12}{20} \cdot \frac{8}{20} = \frac{16}{25} = 0.64.$$

Note that if the third player also spins 60 percent, then she will choose to spin again *as we analyzed previously!* (This is the whole idea of backwards induction – we're figured out in advance how the third player will respond to any action, so we don't need to think about it again.) So, she will only tie if she spins 60 cents total on both spins. The probability of this outcome is

$$\frac{11}{20} \cdot \frac{1}{20} = 0.0275,$$

the probability of spinning less than 60 cents on the first spin, and then on the second spin.

So, in conclusion, with probability 0.3325 you will finish in a two-way tie, and with probability 0.0275 you will finish in a three-way tie. Your winning probability is therefore

$$\frac{1}{2} \cdot 0.3325 + \frac{1}{3} \cdot 0.0275 = .175\ldots$$

Not very good.

Suppose then you spin again. This looks pretty good, right? With probability 0.4 you will improve your score and not be stuck in a tie with the first player.

With probability 0.05 each, you will improve your score to $60 + 5n$ for each of $n = 1, 2, 3, 4, 5, 6, 7, 8$. Your opponent will spin again only if she doesn't match your score. In each case, her winning probability without a tiebreaker is

$$\frac{8-n}{20} + \frac{n+11}{20} \cdot \frac{8-n}{20} = \frac{(8-n)(n+31)}{400}.$$

Her tying probability is

$$\frac{1}{20} + \frac{n+11}{20} \cdot \frac{1}{20} = \frac{n+31}{400}.$$

So her total winning probability is

$$\frac{(8-n)(n+31)}{400} + \frac{n+31}{800} = \frac{-2n^2 - 45n + 527}{800}.$$

That probably looks very strange. For each n from 1 to 8, this is (to three decimal places)

$$0.6, 0.536, 0.468, .393, .315, .231, .142, .048,$$

and so your winning probability is

$$0.4, 0.464, 0.534, .607, .685, .769, .858, .952.$$

To compute your winning probability, multiply each of these numbers by 0.05 and add the results. (Equivalently, take their average and multiply by 0.4 – your probability of not busting.)

We get a winning probability of 0.26. It's **not very good** – you have a probability of only 40% of not busting and even then you could lose.

Let's recap. We have concluded: As the second player, if the first player spins 60 cents, you should spin again if you get less than 60 cents (obvious) or if you tie (less obvious). Also, if you spin 65 cents or more, you should keep it (again not obvious).

8.1.3 Player 2 – In general

We now *mostly* understand what Player 2 should do. We can recap what we have concluded:

- If you spin less than the first player, then obviously you should spin again.
- If you tie the first player at sixty cents, then you should spin again. Also, if you tie the first player at less than 60 cents, then you should still spin again (by comparison with the 60 cent case).

  What if you tie with more than 60 cents? This is a computation we have not done yet. We could similarly work out the smallest amount of money at which you should hold rather than spin again. This is a computation like the one above: we would do the same for 65 cents, 70 cents, 75 cents, ...
- If the first player spins 60 cents, and you surpass her, you should hold.

  It also follows that if the first player spins *more than* 60 cents, and you surpass her, you should still hold.

  Finally, we can conclude that if the first player spins less than 60 cents, and you get at least 65 cents, you should hold. This is equivalent to the case where the first player spun 60 cents. Since you spun more, you have knocked her out of the competition and the only question is whether the third player can beat you.

It remains to consider: if you spin 60 cents or less on your first spin, and it is more than the first player, should you spin again? Somes the answer is clearly yes. Suppose for example that the first player spins twice and gets a nickel each time, for a total of 10 cents. On your first spin you get 15 cents. Then it's fairly clear that you should go again, right? So the question is *when* we should spin again – we just need to compute the cutoff, and we're done.

8.2 Player 1

Finally, what should Player 1 do? Suppose she spins 55 cents. Should she keep it or go again? We can't answer this question completely yet, since we *haven't analyzed Player 2's complete strategy.*

But it's not *so* hard. Suppose she spins 60 cents. Then we have computed Player 2's optimal strategy in every case, and so eventually we can compute Player 1's winning probability if she stays. Similarly we can compute Player 1's winning strategy if she spins again.

Now if we conclude that she should hold on 60 cents, then we would also conclude that she should hold with everything *higher*. Conversely, if we conclude that she should spin again on 60 cents, then we would also conclude that she should spin again with anything *lower*. One feels intuitively that the cutoff point should be around here.

Is this question hard? It sort of feels so. After all, we didn't give a complete answer.

But I want to argue that this question is not, in fact, *so* hard. We gave a complete description of how to solve it in every possible case, and so all we need for a complete solution is the time and willpower to finish.

Or..... a computer. What we described is an *algorithm* for completely solving the question: we break it up into a lot of small steps, and we are guaranteed that following them will eventually yield a complete solution. This is what computers excel at!

This turned out to be worth a research paper. In the first spot, spin again with 65 cents or less:

`http://fac.comtech.depaul.edu/rtenorio/Wheel.pdf`

8.3 Contestant's Row

In the following game, a pie is on a table and three players divide it up. The first player takes any amount of pie, including the entire pie. The second player then takes any amount of pie from *either* the table *or* the first player. The third player then takes any amount of pie from *one* of the first two players, *or* from the table.

How should the players play, if they want to get as much pie as possible?

Before we begin, note that *the last player is definitely at an advantage!* This is because she can take the pie of any other player. Either the portion left over for her is the largest, or she can take the largest piece. In any case she always ends up with more (or as much) pie than the others.

Finally, to simplify matters, we're going to ignore ties in everything that follows. If you leave a tie for another player, then you could have taken just a *tiny* bit less pie and broken the tie – so the other player will definitely not want to take your pie. So we will assume that you always do this.

- We begin, as before with the third player. She picks one piece of pie – either the first player's, the second player's, or whatever is left on the table – and takes all of it. She has no reason to share, and she chooses whichever is largest.

- The second player wants to take as much pie as possible while ensuring that it is not in the third player's interest to take it.

  If the first player took a third or less of the pie, then the second player takes a sliver less than half of the remaining pie. Then, the remaining pie will be the largest piece, the third player will take it, and the second player will end up with nearly as much.

  If the first player took more than a third, but less than half of the pie, then the second player should take a sliver less than the first player took. Then less than a third of the pie will be left and the third player will take the first player's pie.

  If the first player too more than half, but less than two thirds, of the pie, then the second player should take all of the remaining pie. The third player will then take the first player's p Finally, if the first player too more than two thirds (or all) of the pie, the second player should take slightly less than half of the first player's pie. The third player will then take the remainder of the first player's pie.

- Finally, what is the best strategy for the first player? We can see that if she takes more than a third of the pie, it will be taken by the second or third player. So the first player should take slightly less than a third, and leave the rest for the other two players. This leaves both players essentially equally well off.

This game is essentially a model for **Contestant's Row** on The Price Is Right. We introduced this game earlier; see for example the following clip:

`https://www.youtube.com/watch?v=TmKP1a03E2g`

How does the pie game resemble The Price Is Right? And how is it different?

- The resemblence comes from imposing a random model on the range of possible prices. With the scuba gear, the contestants bid 750, 875, 500, and 900, and the actual price was 994.

  So we might assume for example that (1) the price of the scuba gear was between $500 and $1,200; that (2) that the price is equally likely to be any of these; that (3) all of the players know this; and that (4) all of the players know that all of the players know this. (And, all of the players know that all of the players know that all of the players know this, and so on.)

  And, as always, we have to assume that all of the players are math experts and play rationally in their own self-interest. This is *demonstrably* false – the fourth player should *always* bid either exactly $1 more than some other player, or exactly $1. But most contestants don't do this. This is kind of like going last in the pie game, and taking someone else's pie – but taking only some of it. In our cutthroat, dog-eat-dog worldview, there's no reason to take less than all of it.

  Note that *none of our assumptions* are totally realistic. That is the price for developing a mathematical model. Any of our assumptions can be questioned. But we have to make assumptions to tackle this as a math problem.

So, in our model, the pie is the price range, and the amount of pie each player has is the range of guesses that their bid has covered. Our four contestants end up with the following ranges: 750 to 874, 875 to 899, 500 to 749, and 900 to 1200. The last player has done the best, the third player has done almost as well, and the second player has had his pie taken from him.

- The Price Is Right is *discrete*, where our pie game was *continuous*. You can divide a piece of pie, no matter how tiny, into still smaller pieces. But on The Price Is Right, the minimum interval is one dollar.

  This doesn't make a big difference, but it does make a difference.

- Related to the above, you can't literally steal someone else's bid – this is like saying you have to leave someone with a tiny piece of pie. You *do* see on the show one contestant bidding one over an earlier contestant, and the earlier contestant being exactly right. (Indeed, contestants get a cash bonus if they guess the amount on the spot.)

- If all of the contestants bid too high, then they don't all lose. They all get to bid again.

  So, in our pie model, this is like saying that if there is any unclaimed pie, the players get to play again for the leftovers. Or (essentially equivalently), what matters is not how much pie you get – but rather that you get *more* than your competitors!

We could tweak our pie model. But we don't want to think of mathematical models as being 'right' or 'wrong' per se – rather, we try to make them fairly accurate, and design them to capture the essential elements of the game.

9 Special Topics

In this section we treat some unusual mathematical topics which come up in game shows. (During the course, we covered these just before the midterm; it was intended that students see these topics but not be expected to master them.)

9.1 Divisibility Tests

The following clip illustrates the Price Is Right game of **Hit Me**.

`https://www.youtube.com/watch?v=n5dZcIq7fIk&t=189s`

Game Description (Hit Me (The Price Is Right)): The contestant plays a game of blackjack against the dealer, where the objective is to get a total of 21 without going over. The dealer plays as in ordinary blackjack: it deals two cards at random, and if it has a total of 16 or less it keeps drawing cards until it is over 16.

The contestant is shown six prizes along with six prices, each of which is $n\times$ the actual price of the item. Behind each prize is a card worth n. The contestant chooses prices one at a time and her hand is made up of these cards.

One of the prices will always be exactly right (so $n = 1$, and the card is ace, which in blackjack you may count as eleven), and one of them will be ten times the right price. If the contestant picks these two prizes first, she gets a blackjack (21) and wins no matter what the dealer has. Otherwise, she still has some opportunities to win.

In the clip, the contestant is shown the following prizes and prices: some kind of joint cream for \$5.58; toothpaste for \$14.37; some fragrance for \$64.90; a six-pack of juice for (???? – poor camera work); some calcium supplements for \$76.79; and some denture adhesive for \$27.12.

We now ask which cards these prizes might hide. And for this we review the *divisibility tests* from number theory:

- A number is divisible by 2 iff its last digit is.
- A number is divisible by 3 iff the sum of its digits is.
- A number is divisible by 4 iff its last two digits are.
- A number is divisible by 5 iff its last digit is.
- A number is divisible by 6 iff it is divisible by both 2 and 3.
- There are divisibility tests for 7, but it is probably easier to just try dividing by 7 in your head.
- A number is divisible by 8 iff its last three digits are.
- A number is divisible by 9 iff the sum of its digits is.
- A number is divisible by 10 iff its last digit is 0.

No, ‘iff’ is not a typo. The word **iff** is mathematical short-hand for **if and only if**, describing a **necessary and sufficient condition**. For example, if the sum of a number’s digits is divisible by 3, then the number is divisible by 3. If the sum of a number’s digits is *not* divisible by 3, the number is *not* divisible by 3.

In mathematics we are always very happy when we have necessary and sufficient conditions. Sometimes we have only one or the other. For example, if a number ends in the digit 6, then we know it is divisible by 2, but vice versa. Conversely, if we want to test if a number is divisible by 8, we can just apply the divisibility test for 4. If the number is indeed divisible by 4 then we have more work to do, but if it’s not then it can’t be divisible by 8 either.

So we can use these to figure out what prices are divisible by what.

- 558 is divisible by 2, 3, 6, and 9. Pretty obviously the joint cream is not 62 cents, but it could well be \$2.79 and so this one is a little bit tricky to guess.

  So, the card could be any of the ace, two, three, six, or nine, with the ace or the two more likely.

- 1437 is divisible by 3 (only). It looks like the toothpaste is \$4.79.

- 6490 is divisible by 2, 5, and 10. If the price of the juice does not end in a ten, and we know that one of the cards is a ten (which it always is), then we know this has to be the ten.

- We have no idea what card the juice hides, because the camera operator is incompetent.

- 7679 is divisible by 7 (only). The cheap way to see this is to eliminate 3 and 9 immediately; it's not even, it's not divisible by 5, so 7 is the only thing that's left unless we believe that the supplements cost this much money.

  We can also notice that 7700 is divisible by 7; now subtract 21.

- 2712 is divisible by 2, 3, 4, 6, and 8. It's difficult to guess the actual price.

We can see if that if you are willing to do some arithmetic in your head, you can do quite well in this game!

9.2 Bonkers, Gray Codes, and Mathematical Induction

Here is a clip of the Price Is Right game of **Bonkers**:

`https://www.youtube.com/watch?v=3EqBci6OQNg`

Game Description (Bonkers (The Price Is Right)): The contestant is shown a prize whose price is four digits. She is then shown a board with a four digit price for the item. Each digit is wrong, and there are spaces to put paddles above and below each digit.

She must guess whether each digit is too high or too low, by placing paddles in the appropriate location and hitting a button (after which she gets the prize if her guess is right, and buzzed if it is wrong). She has thirty seconds and may guess as many times as she is physically able to.

She does win the prize, but she only gets off four guesses and wins at the last second. Her strategy leaves much to be improved upon. Here is a contestant who puts on a much better show:

`https://www.youtube.com/watch?v=iZzBu5K_aBA`

We can ask: what's the optimal strategy? Is there an efficient, and easily remembered, algorithm to go through every possibility?

The best we can possibly do is to move 19 paddles. You make some starting guess (you need to move 4 paddles for this), and then there are 15 more possible guesses. (The total number of possibilities is $2^4 = 16$.) We will achieve this, and more.

Theorem 9.1 *Suppose you play a game of Bonkers with* n *digits* ($n = 1, 2, 3, 4, \ldots$)*, and the* n *paddles are arranged in any guess. Then it is possible to cycle through all remaining* $2^n - 1$ *guesses by moving only one paddle at a time – so* $2^n - 1$ *paddle moves.*

We are interested in the case $n = 4$ – but it is actually ***easier*** to prove this for **all** n **at the same time!** The algorithm is beautifully simple:

- Step 1. Go through all possibilities for the first $n - 1$ paddles. This requires $2^{n-1} - 1$ moves.
- Step 2. Move the last paddle. This requires 1 move.
- Step 3. Again go through all possibilities for the first $n - 1$ paddles. This requires $2^{n-1} - 1$ moves.

The total number of moves required is

$$(2^{n-1} - 1) + 1 + (2^{n-1} - 1) = 2 \cdot 2^{n-1} - 1 + 1 - 1 = 2^n - 1.$$

So in other words, **if** we can solve Bonkers with one paddle, we can solve it with two. **If** we can solve Bonkers with two paddles, we can solve it with three. **If** we can solve Bonkers with three paddles, we can solve it with four. And so on, forever. The solution with one paddle is trivial ($2^1 - 1 = 1$, and we simply move the paddle from one slot to the other), but this is the building block that sets off a chain reaction, allowing us to solve Bonkers for any number of paddles. So if the price was twenty digits, we could win the game within 1,048,595 moves – twenty to fix the paddles initially, and $2^20 - 1 = 1048575$ to iterate through the remaining guesses.

This process of reasoning is known as **induction** by mathematicians, and **recursion** by computer programmers. In each case it is an extraordinarily powerful tool.

So let's see how it works in practice:

- Bonkers with one paddle: Move the following paddles in order: 1. If our starting position is T (T for top and B for bottom), then this results in the following sequence of positions: T, B.
- Bonkers with two paddles: Move the following paddles in order: $1, 2, 1$. If our starting position is TT, then this results in the following sequence of positions: TT, BT, BB, TB.

- Bonkers with three paddles: Move the following paddles in order: $1, 2, 1, 3, 1, 2, 1$. If our starting position is TTT, then this results in the following sequence of positions: TTT, BTT, BBT, TBT, TBB, BBB, BTB, TTB.
- Bonkers with four paddles: Move the following paddles in order: $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1$. If our starting position is TTT, then this results in the following sequence of positions: TTTT, BTTT, BBTT, TBTT, TBBT, BBBT, BTBT, TTBT, TTBB, BTBB, BBBB, TBBB, TBTB, BBTB, BTTB, TTTB.

We can see the recursive structure of our solutions more clearly in the instructions than in the resulting sequence of paddles. For example, if we write (sequence for 3) for $1, 2, 1, 3, 1, 2, 1$, then the last sequence for 4 is (sequence for 3), 4, (sequence for 3). Similarly if we call that whole thing (sequence for 4), the sequence for 5 is (sequence for 4), 5, (sequence for 4).

The same pattern can be seen in the tick lengths on many rulers!

These sequences of T's and B's are known as **binary Gray codes** and have applications in electrical engineering.

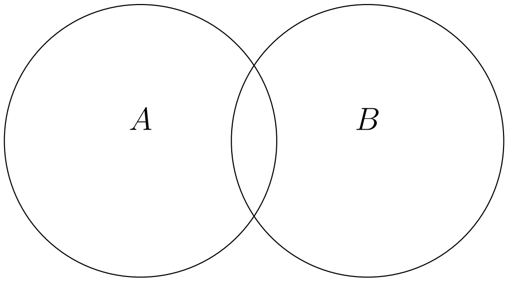
9.3 Inclusion-Exclusion

The principle of **inclusion and exclusion** is a bit difficult to explain, but it is a powerful one. It does not really come up when analyzing game shows, but it illustrates many of the same principles. It is most easily explained by example.

For example, suppose we want to count elements in the union $A \cup B$, where A and B are any two sets. Then, we have

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

This is illustrated by the following Venn diagram. Everything in $A \cap B$ was counted twice – once for A, once for B, so we need to subtract it once to make sure it wasn't double counted.



Example 9.2 *How many integers between* 1 *and* 100 *are divisible by either* 2 *or* 3*?*

Solution. There are 50 integers in the set divisible by 2, and 33 divisible by 3. (3×1 through 3×33.) An integer is divisible by both 2 and 3 if and only if it is divisible by 6, and there are 16 of these.

So the count is

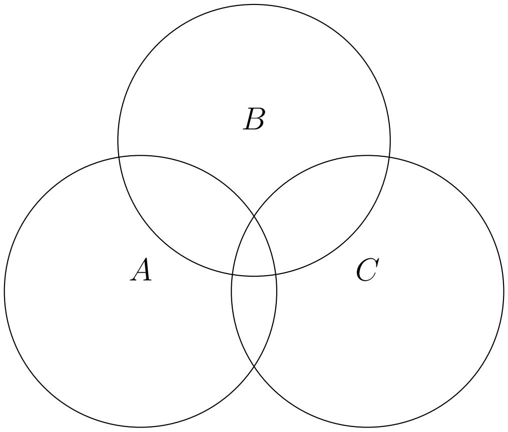
$$50 + 33 - 16 = 67.$$

You can check it! Another way to count the same: an integer n is divisible by 2 or 3 if its remainder after division by 6 is 2, 3, 4, or 0. So, four out of every six. In the first 96 integers, there are 16 groups of six and exactly 64 integers in this range that we want to count. Finally, out of the last four integers (97, 98, 99, and 100) there are fhree we want to count. So 67 total.

If we have *three* sets A, B, and C, then the rule is

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Here is a Venn diagram.



The formula is probably not obvious, but you can check it from the diagram. For example, if an element is in A, not B, and not C, then the number of times it is counted is

$$1 + 0 + 0 - 0 - 0 - 0 + 0 = 1.$$

The terms are in the same order as in the formula above. It appears in only the $N(A)$ term on the right!

If an element is in A and B, but not C, then the number of times it is counted is

$$1 + 1 + 0 - 1 - 0 - 0 + 0 = 1,$$

and if an element is in all three then the number of times it is counted is

$$1 + 1 + 1 - 1 - 1 - 1 + 1 = 1.$$

The other cases are exactly analogous. So, each element is counted exactly once, unless it is in none of the sets.

The general **principle of inclusion-exclusion** says that this process works with any finite number of sets. Given sets A_1, A_2, through A_k,

- Consider each of the sets, and sum all of their sizes. Then,
- Consider the double overlaps, and sum the sizes of all the double overlaps. Subtract these. Then,
- Consider the triple overlaps, and sum all their sizes. Add these. Then,
- Consider the quadruple overlaps, and sum all their sizes. Subtract these. Then, ...
- And so on. Keep going, alternating addition and subtraction until you've counted the overlap of all of the sets. (You will count it negative if there are an even number of sets, and positive if there are an odd number of sets.)

9.3.1 The umbrella problem

We consider the following problem:

The Umbrella Problem. One hundred guests attend a party. It is raining, and they all bring umbrellas to the party. All of their umbrellas are different from each other.

At the end of the party, the host hands umbrellas back to the guests at random. What is the probability that *nobody* gets their own umbrella back?

This was asked as a probability question, but we will reframe it as a counting question. We consider the number of ways to give umbrellas out to the party guests. This is essentially the same as a *permutation* of the umbrellas, so there are 100! possibilities. This number is equal to

93326215443944152681699238856266700490715968264381621468592963895217599993229915608941463976156518286253697920827223758251185210916864000000000000000000000000.

We just have to count how many of them don't involve giving anyone their own umbrella! No problem, right?

We'll do the opposite count, and count how many involve giving *at least* one person their own umbrella. We do this using inclusion-exclusion. A_1 is the set of ways to distribute the umbrellas, with person #1 getting their own umbrella. A_2 is the set of ways to distribute the umbrellas, with person #2 getting their own umbrella. And so on, there are a hundred sets.

Note that, when counting A_1, we don't have to worry about whether other people get their umbrella or not! This is much easier. If we needed to count the ways to distribute the umbrellas, with person #1 and *only person #1* getting their umbrella, this would be harder. (It would be like the mani problem we're solving here.) The nice thing is that we have formulated the problem so we never have to worry about who *doesn't* get their own umbrella, only who *does*.

We count $N(A_1 \cup A_2 \cup \cdots \cup A_{100})$ and then subtract it from that big number. We do this using inclusion-exclusion.

- $N(A_1)$ is just 99!. We give person #1 their own umbrella, and distribute the other umbrellas any which way.

  $N(A_2)$ through $N(A_{100})$ are also each 99!, for the same reason.

  So, the total number added in this step is

  $$99! \times 100 = 100!.$$

- $N(A_1 \cup A_2)$ is 98!. We give the first two people their own umbrella, and distribute the others however.

  How many sets are there like this? Exactly the number of ways to choose two people out of 100, which is $C(100, 2) = \frac{100!}{98!2!}$. So the total number subtracted in this step is

  $$98! \times \frac{100!}{98!2!} = \frac{100!}{2!}.$$

- $N(A_1 \cup A_2 \cup A_3)$ is 97!, as before. The number of sets like this is $C(100, 3) = \frac{100!}{97!3!}$, and the number *added* in this step is

  $$97! \times \frac{100!}{97!3!} = \frac{100!}{3!}.$$

- The pattern continues. For the four-fold intersections we subtract

  $$96! \times \frac{100!}{96!4!} = \frac{100!}{4!},$$

  and then we add $\frac{100!}{5!}$, subtract $\frac{100!}{6!}$, and so on. The very last step is subtracting $\frac{100!}{100!}$ – the one way in which we can give everyone their correct umbrella!

So the total number of ways to distribute the umbrellas with *at least one person* getting their umbrella is

$$100! - \frac{100!}{2!} + \frac{100!}{3!} - \frac{100!}{4!} + \cdots - \frac{100!}{100!}.$$

The number of ways to distribute the umbrellas with *nobody* getting their umbrella is 100! minus this, or

$$100! - 100! + \frac{100!}{2!} - \frac{100!}{3!} + \frac{100!}{4!} - \cdots + \frac{100!}{100!},$$

which we can rewrite as

$$100!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!}\right),$$

and so upon dividing by 100! (which was the *total* number of ways to distribute the umbrellas we see that the probability that no one gets their umbrella is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!}.$$

This is a pretty good answer! But we can do better if we know some calculus. Calculus tells us that the **Taylor series expansion** for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

and so plugging in e^{-1} we get

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!} - \frac{1}{101!} + \cdots$$

If we truncate after $\frac{1}{100!}$ then we get **exactly our umbrella probability**, and also by the **alternating series test** we make an error less than $\frac{1}{101!}$, which is very **VERY** small – less than one over the **GIANT** number above, and so the probability that no one gets their umbrella is, within an error bounded by $\frac{1}{101!}$, equal to

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = 0.36787944117144232159552377016146086745 \cdots$$

9.3.2 Switcheroo

The game **Switcheroo** is illustrated in the following clip.

`https://www.youtube.com/watch?v=nvSMVAuGpAE`

Game Description (Switcheroo – The Price Is Right): The contestant is shown five prizes – four very small prizes and a car. The contestant is also shown the price of each with the tens digits removed. The tens digits of the prizes are all different from each other, and the contestant is given five blocks with the tens digits written on them.

The contestant must match the blocks to the removed tens digits. She has thirty seconds to do this, after which she is shown how many prizes she has matched correctly (but not *which* prizes she has matched correctly). She may then switch around the blocks if she likes. She wins all prizes which she prices correctly.

We will ask the following question: **A contestant has no idea what any of the prices are, but otherwise plays optimally. With what probability does she win the car?**

Since she has no idea what the prices are, she just places the blocks randomly in the first round (any placement is as good as any other). Her choice of strategy depends on how many she gets right on the first round:

- If she gets 2, 3, or 5 correct then this is better than expected and she should stick with her guess. Her probability of winning the car is $\frac{2}{5}$, $\frac{3}{5}$, or 1 respectively.

  (Note that there is no way to get exactly four correct, or equivalently, exactly one wrong. If one prize's block is in another prize's slot, then her guesses for *both* prizes must be wrong.)

- If she gets 1 correct then she is indifferent to switching or leaving everything in place. There is a $\frac{1}{5}$ probability that her one correct item is the car, and a $\frac{4}{5}$ probability that it is one of the other items – in which case each of the remaining numbers is equally likely to be the correct price for the car.

- If she gets none correct, then she should switch. This is *better* than getting exactly one correct because she got some reliable information: her guess for the car is wrong. So she picks one of the other numbers and wins with probability $\frac{1}{4}$.

So we have to figure out the probabilities of each of these outcomes on the first round! There are $5! = 120$ ways to place the blocks, and so all of these probabilities will be fractions with 120 in the denominator. In our analysis we will label the prizes A, B, C, D, and E.

- *Five right.* There is exactly one way.

- *Three right.* First we ask: in how many ways can the contestant get A, B, and C right and D and E wrong? One: ABCED.

  So the number of ways to get exactly three right is $C(5,3) = 10$ – the number of subsets of three of the five prizes.

- *Two right.* First we ask: in how many ways can the contestant get A and B right and C, D, and E wrong? There are two: ABDEC and ABECD.

  So the number of ways to get exactly three right is $C(5,3) \times 2 = 10 \times 2 = 20$: there are ten ways in which to choose which subset she gets right, and for each, two ways to screw the rest up.

- *None right.* This is the umbrella problem!! The answer is

$$5! - 5! + \frac{5}{2!} - \frac{5}{3!} + \frac{5}{4!} - \frac{5}{5!} = 44.$$

- *One right.* By process of elimination,

$$120 - (1 + 10 + 20 + 44) = 120 - 75 = 45.$$

  Alternatively, there are five ways to choose one prize to get right, and

$$4! - 4! + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 9$$

  ways to mix up the rest, and $5 \times 9 = 45$.

So her probability of winning in the end is

$$\frac{1}{120} \times 1 + \frac{10}{120} \times \frac{3}{5} + \frac{20}{120} \times \frac{2}{5} + \frac{45}{120} \times \frac{1}{5} + \frac{44}{120} \times \frac{1}{4} = \frac{7}{24}.$$

10 Review

Here we briefly review the main ideas of the course and propose some sample questions relevant to each. (This is *not* a comprehensive listing of every topic covered.)

Probability and counting We formally defined probability in terms of *sample spaces* and *events*. These were subject to the *addition* and *multiplication* rules. The former said that probabilities for disjoint events add; the latter that probabilities for sequential events multiply.

We also supplemented this material with some material on permuations and combinations – you should remember the formulas for those. These help with counting sizes of events and sample spaces.

Good sample questions involve coins, dice, and cards. You are dealt two cards. What is the probability that (1) they are of the same suit; (2) they are each ten or higher; (3) they could possibly fit into a five-card straight; etc. (Compose your own!) You toss three dice. What is the probability the sum is odd? Even? At least twelve? Bigger than 14 or smaller than 5? You flip ten coins. What is the probability that they all come up heads? Half of them?

Probability questions also came up in various game shows. On The Price Is Right, Rat Race, Let 'Em Roll, Squeeze Play, Switcheroo, 3 Strikes, Spelling Bee, and Plinko (among others) provide lots of probability questions. Watch an episode, start to finish, and see what you can come up with. You can also come up with interesing probability questions watching Deal or No Deal: what is the probability that the contestant will have eliminated the two most valuable briefcase by the time the bank's first offer comes in?

Finally, poker was an excellent source of probability questions.

Expected value. Make sure you understand how expected value works. Expected value comes up in poker, in game shows like Let's Make a Deal and Deal or No Deal, and pretty much every scenario where probability is relevant. Here again you can compose your own questions. You toss three dice, and get a dollar for every six you roll. Alternatively, you get a dollar if at least two dice are the same. What is the value of playing such a game?

Remember the rule of *linearity of expectation*, and review its applications. The idea is that *expected values add.* For example, if you get a dollar for every six you roll in three dice, you do *not* need to compute the probabilities of rolling zero, one, two, or three sixes. Just compute the expected value of one die, and multiply by three.

We also introduced conditional probability. Make sure you understand the definition and why it is true. Review the Monty Hall Problem and its variants (and related games like Barker's Markers). And be sure you understand how to use Bayes's theorem, either in the form of the formula or in terms of reasoning via probability trees.

Strategic game theory. We covered this *very* lightly (it is easily worth an entire undergraduate course). Understand how these are set up and how a payoff matrix corresponds to a game. You should also be able to find the Nash equlibrium in a game with two choices for each player. This might be a pure or mixed strategy. (Try finding the Nash equilibria of all the games we discussed, and thinking up your own games.)

Backwards induction. There were no exercises on this, and any exam problems on this will be relatively easy. Note also that we did some backwards induction problems before introducing it per se. For example, Punch-a-Bunch is very much backwards induction.

11 Project Ideas

Part of the course requirements is a term project: study a game or game show in depth, write a paper analyzing it, and give a presentation in class.

Here are some ideas. Of course, feel free to come up with your own.

- **Deal or No Deal**: This is an easy to understand game from the contestants' point of view. What about the producers? How does the bank determine its offers?

  Watch a bunch of episodes of the show and write down what happens. Attempt to determine some sort of formula that predicts what the bank will offer.

- **Press Your Luck**: One interesting project would be to investigate the patterns behind the show, just as Michael Larson did. Watch old YouTube videos, and hit freeze frame a lot! Try to describe the patterns, and see if you could win $100,000 too.

- **Switcheroo**: Here is a fascinating, and deep Price Is Right game:

  `https://www.youtube.com/watch?v=nvSMVAuGpAE`

  Try to figure out the optimal strategy. You will have to assume that the contestant has *some* idea how much the small prizes cost, but very imperfect information.

- **Race Game**: A somewhat easier Price is Right game. Here is a clip:

  `https://www.youtube.com/watch?v=CkqZkqeNyKU`

  You might try to figure out the best strategy, assuming the contestant has no idea how much the prizes cost.

  This is somewhat similar to the game *Mastermind* (see the Wikipedia page). But don't neglect the fact that some prizes are closer to the lever than others!

- **Poker**: If you enjoyed the poker discussion, you might want to dig deeper. I recommend reading at least the first of Harrington's books, watching some poker tournaments online, and then trying to analyze what happened.

12 Review of Games, Links, and Principles

(p. 12) **The Addition Rule (1).** Suppose E and F are two *disjoint* events in the *same sample space* – i.e., they don't overlap. Then

$$P(E \text{ or } F) = P(E) + P(F).$$

(p. 13) **The Multiplication Rule.** The multiplication rule computes the probability that two events E and F **both** occur. Here they are events in **different** sample spaces.

The formula is the following:

$$P(E \text{ and } F) = P(E) \times P(F).$$

It is not always valid, but it is valid in either of the following circumstances:

- The events E and F are *independent*.
- The probability given for F assumes that the event E occurs (or vice versa).

(p. 15) **Michael Larson.** Here is a bit of game show history. The following clip comes from the game show Press Your Luck on May 19, 1984.

`https://www.youtube.com/watch?v=UzggoA4lLwk`

(p. 16) **Game Description** (Card Sharks): Each of two contestants receives a lineup of five cards. The first is shown to each contestant, and a *marker* is placed on the first card. The objective of each **round** is to reach the last card.

A **turn** by the contestant consists of the following. She starts with the (face-up) card at the marker, and may replace it with a random card if she chooses. She then guesses whether the next card is higher or lower, which is then revealed.

If is the last card and her guess is correct, she wins the round. Otherwise, she may keep guessing cards for as long as she likes untill one of three things happens: (1) she guesses the last card correctly, and wins; (2) she guesses any card incorrectly, in which case the cards she has guessed are all discarded and replaced with new cards (face down); (3) she chooses to end the turn by moving her marker forward to the last card guessed correctly.

The **round** begins with a trivia question (I don't describe the rules for that here), and the winner gets to take a turn. If this turn ends with a freeze, the contestants go to another trivia question; if it ends with a loss, the other contestant takes a turn.

(p. 17) Here[22] is a typical clip:

`https://www.youtube.com/watch?v=bUvOCRU6t5o`

(p. 20) This video[23] illustrates a playing of the Price Is Right game **Ten Chances**:

`https://www.youtube.com/watch?v=iY_gmGcDKXE`

(p. 20) **Game Description** (Ten Chances (The Price Is Right)): The contestant is shown a small prize, a medium prize, and a large prize. She has ten chances to win as many prizes as she can.

The price of small prize has two numbers in it, and the contestant is shown three different numbers. She then guesses the price of the first prize. She takes as many chances as she needs to.

Once she wins the small prize, she attempts to win the medium prize. The price of the medium prize has three numbers in it, and the contestant is shown four.

Finally, if she wins the medium prize, she attempts to win the car. Its price has five numbers in it, and the contestant is shown these five.

(p. 22)

Definition 12.1 *Let T be a* **string**. *For example,* 01568 *and* 22045 *are strings of numbers, ABC and xyz are strings of letters, and* $\otimes - \oplus$♣☠ *is a string of symbols. Order matters:* 01568 *is not the same string as* 05186.

A **permutation** *of T is any reordering of T.*

(p. 22)

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[22] *Summary of the clip*: (**Please note.** The trivia questions are off-color and arguably sexist. This is unfortunately common on this show.) The contestants are Royce and Cynthia. Cynthia wins the first trivia question. Her initial card is a king. She keeps it and guesses lower; the second card is a two. She guesses higher; the third card is a nine. She freezes on position three.

Royce wins the next trivia question. His initial card is an eight; he changes it and gets a four. He guesses higher; the second card is a six. He guesses higher; the third card is a nine. He freezes on position three.

Royce wins the next trivia question. He starts on position three and chooses to replace the nine, and gets a three. He guesses higher; the fourth card is a five. He guesses higher; the fifth card is a king and Royce wins the round.

[23] *Summary of the clip*: She plays Ten Chances for a pasta maker, a lawnmower, and a car. The digits in the pasta maker are 069, and she guesses the correct price of 90 on her second chance. The digits in the mower are 0689, and she guesses the correct price of 980 on her third chance. (Her third chance overall; she took only once to win the mower.) The digits in the car are 01568, and she guesses the correct price of $16,580$ on her first try (and wins).

Barker then hides beyond the prop ... and, uh, (**please note**) the contestant violates his personal space.

Proposition 12.2 *Let T be a string with n distinct symbols. Then there are exactly $n!$ distinct permutations of T.*

(p. 28)

Definition 12.3 *Consider a random process whose outcome can be described as a real number. Suppose that the possible outcomes are $a_1, a_2, \dots a_n$, which occur with respective probabilities $p_1, p_2, \dots, p_n$. Then the* **expected value** *of this process is*

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k.$$

(p. 29) **Game Description** (Wheel of Fortune, Simplified Version): The contestants play several rounds where they try to solve word puzzles and win money. (The contestant who has won the most money then gets to play in a bonus round.)

The puzzle consists of a phrase whose letters are all hidden. In turn, each contestant either **attempts to solve the puzzle** or **spins the wheel**. If the contestant attempts to solve, he states a guess; if is correct, he wins all the money in his bank, and if it is wrong, play passes to the next player.

The wheel contains lots of spaces with various dollar amounts or the word 'bankrupt'. When the contest spins, the wheel comes to rest on one of these spaces. If 'bankrupt', the contestant loses all his money from this round and play passes to the next contestant. Otherwise, the contestant chooses a letter. If that letter appears in the puzzle (and has not yet been guessed), then each of these letters is revealed and the contestant wins the amount of money on his space for each time it appears. If the letter does not appear, the contestant wins nothing and play passes to the next contestant.

(p. 29) Consider the episode of Wheel of Fortune shown in this clip:

`https://www.youtube.com/watch?v=A8bZUXi7zDE`

Robert wins the first round in short order. After guessing only two letters (and buying a vowel) he chooses to solve the puzzle. Was his decision wise?

(p. 31) **Game Description** (Punch-a-Bunch (The Price Is Right)): The contestant is shown a punching board which contains 50 slots with the following dollar amounts: 100 (5), 250 (10), 500 (10), 1000 (10), 2500 (8), 5000 (4), 10,000 (2), 25,000 (1). The contestant can earn up to four punches by pricing small items correclty. For each punch, the contestant punches out one hole in the board.

The host proceeds through the holes punched one at a time. The host shows the contestant the amount of money he has won, and he has the option of either taking it and ending the game, or discarding and going on to the next hole.

(p. 31) Here is a typical playing of Punch-a-Bunch:

`https://www.youtube.com/watch?v=25THBiZNPpo`

(p. 32) Here is a typical clip from Who Wants To Be a Millionaire:

`https://www.youtube.com/watch?v=sTGxOqp3qB8`

(p. 33) **Game Description** (Who Wants to be a Millionaire?): The contestant is provided with a sequence of 15 trivia questions, each of which is multiple choice with four possible answers. They are worth an increasing amount of money: 100, 200, 300, 500, and then (in thousands) 1, 2, 4, 6, 16, 32, 64, 125, 250, 500, 1000. (In fact, in this epsiode, the million dollar question was worth $2,060,000.)

At each stage he is asked a trivia question for the next higher dollar amount. He can choose to answer, or to not answer and to keep his winnings so far. If he answers correctly, he goes to the next level. If he answers incorrectly, the game is over. At the $1,000 and $32,000 level his winnings are protected: he is guaranteed of winning at least that much money. Beyond that, he forfeits any winnings if he ventures an incorrect answer.

He has three 'lifelines', each of which may be used exactly once over the course of the game: '50-50', which eliminates two of the possible answers; 'phone a friend', allowing him to call a friend for help; and 'ask the audience', allowing him to poll the audience for their opinion.

(p. 36) **Principle of Linearity of Expectation.** Suppose that we have a random process which can be broken up into two or more separate processes. Then, the total expected value is equal to the sum of the expected values of the smaller processes.

This is true whether or not the smaller processes are independent of each other.

(p. 38) The next questions concern the Price is Right game **Let 'em Roll**. Here is a clip:

`https://www.youtube.com/watch?v=g5qF-W9cSpo`

(p. 38) **Game Description** (Let 'em Roll (Price Is Right)):

The contestant has five dice to roll. Each die has $500 on one side, $1,000 on another, $1,500 on a third, and a car symbol on the other three. The contestant rolls all five dice. If a car symbol is showing on each of them, she wins the car. Otherwise, she wins the total amount of money showing. (Car symbols count nothing, unless she wins the car.)

By default, the contestant gets one roll, and may earn up to two more by correctly pricing small grocery items. After each roll, if she gets another roll, she may either keep all the money showing, or set the dice showing 'car' aside and reroll only the rest.

(p. 40) **The multiplication rule for counting.** Suppose that an operation consists of k steps, and:

- The first step can be performed in n_1 ways;
- The second step can be performed in n_2 ways (regardless of how the first step was performed);
- and so on. Finally the kth step can be performed in n_k ways (regardless of how the preceding steps were performed).

Then the entire operation can be performed in $n_1 n_2 \dots n_k$ ways.

(p. 42) **Notation.** Write $P(n, r)$ for the number of r-permutations of a string with n distinct symbols.

We have the following formula:

$$P(n, r) = \frac{n!}{(n-r)!}$$

(p. 43) If we start with a string (or a set) with n distinct elements, then an r-**combination** is a string or r of these elements *where order doesn't matter*, or equivalently a subset of r of these elements.

(p. 43) **Notation.** Write $C(n, r)$ or $\binom{n}{r}$ for the number of r-combinations of an n-element set.

(p. 44)

Theorem 12.4 *We have*

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(p. 45) Here is a video of the Price Is Right game **Plinko**:

`https://www.youtube.com/watch?v=qr7oYqcgsXQ`

(p. 45) **Game Description** (Plinko (The Price Is Right)): The contestant drops up to five chips down a board. (She starts off with one, and can win up to four more by pricing small items.) She drops them down a board which has a lot of pegs and a variety of prizes at the bottom. (The shape of the board **is** relevant, and we will discuss it more in due course.) She hopes to land her chips into a \$10,000 slot in the middle, and the other slots have prizes between zero and \$1,000.

(p. 45) **Pascal's Triangle.** To write down Pascal's Triangle, proceed as follows.

- The top row has a solitary 1 in it.
- Each row has one more number than the previous, with a 1 at each edge. **Each number in the middle of the table is equal to the sum of the two above it.**
- Proceed for as many rows as you like.
- By convention the rows are numbered as follows: the top row is the **zeroth** row. After that, the rows are numbered 1, 2, 3, etc., and the nth row starts with a 1 and an n.

(p. 45) Our idealized version of Plinko is illustrated nicely by the following computer demonstration:

`phet.colorado.edu/sims/plinko-probability/plinko-probability_en.html`

(p. 45)

Proposition 12.5 *The numbers in the nth row of Pascal's Triangle sum to* 2^n.

(p. 46)

Proposition 12.6 *The numbers in the nth row of Pascal's Triangle are* $C(n,0)$, $C(n,1)$, *...,* $C(n,n)$ *in order.*

(p. 46)

Proposition 12.7 *We have* $C(n,r) = C(n,n-r)$ *for all n and r.*

(p. 46)

Proposition 12.8 *The biggest numbers are always in the middle.*

(p. 46)

Proposition 12.9 *We have, for all n and r, that*

$$C(n,r) + C(n,r+1) = C(n+1,r+1).$$

(p. 47)

Proposition 12.10 *You can read off a rule for FOILing from Pascal's Triangle. In particular, you have*

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \cdots + C(n,n)y^n.$$

(p. 47)

Proposition 12.11 *The alternating sum of each row of Pascal's Triangle (after the zeroth) is* 0.

(p. 48)

Proposition 12.12 *If you color all the odd numbers blue and the even numbers red, you will create a familiar pattern called the 'Sierpinski triangle' which is a* **fractal**.

(p. 48)

Proposition 12.13 *Suppose you draw lines through Pascal's Triangle at an angle.*

For example, start at any of the 1*'s on the left. Circle it. Then, go over to the right one and up and right one, and circle that number. Then, again go over to the right one and up and right one and circle that. Keep going until you run out of numbers.*

If you add up all the numbers you circled, you get .....

(p. 48)

Proposition 12.14 *The distribution of Pascal's triangle approaches a nice limit as* $n \to \infty$.

(p. 48) Here is a website which allows you to conduct experiments like this:

`http://www.math.uah.edu/stat/apps/BinomialTimelineExperiment.html`

(p. 65)

`https://www.youtube.com/watch?v=-vRty_kkfgw`

(p. 65)

Definition 12.15 *Let* A *and* B *be events in a sample space* S. *If* $P(A) \neq 0$, *then the* **conditional probability of** B **given** A, *written* $P(B|A)$, *is*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

(p. 66)

`https://www.youtube.com/watch?v=V6gCNW5wFIY`

(p. 66) **Game Description** (One Away – The Price Is Right): The contestant is shown a car and a five digit price for the car. Each digit in the price is off by one – too low or too high. She then guesses the price of the car, one digit at a time.

If her guess is correct, she wins the car. Otherwise, if at least one digit is correct, she is told how many digits she has right and can make corrections as she sees fit.

(p. 67) **The Monty Hall Problem.** Monty Hall, on Let's Make a Deal, shows you three doors. Behind one door is a car, behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to switch to door No. 2?"

Is it to your advantage to switch your choice?

(p. 71)

Theorem 12.16 (Bayes' Theorem) *Suppose that A and B are any two events. Then we have*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

(p. 76) **The Monty Hall Problem – Zonk!** Monty shows you three doors, behind one of which is a car. You pick Door #1, and Monty shows you Door #3, behind which is – the car!

You lose. Zonk.

(p. 82) **Game Description** (Golden Balls (Final Round)): Two players play for a fixed jackpot, the amount of which was determined in earlier rounds. They each have two balls, labeled 'split' and 'steal'. They are given some time to discuss their strategies with each other. Then, they each secretly choose one of the balls and their choices are revealed to each other.

If both choose the 'split' ball, they split the jackpot. If one chooses the 'split' ball, and the other 'steal', the player choosing 'steal' gets the entire jackpot. If both players choose 'steal', they walk away with nothing.

(p. 86) **Game Description** (Jeopardy – Final Jeopardy): Three players come into the final round with various amounts of money. They are shown a category and write down a dollar amount (anything up to their total) that they wish to wager.

After they record their wagers, they are asked a trivia question. They gain or lose the amount of their wager, depending on whether their answer was correct. Only the top finisher gets to keep their money.

(p. 87)

Definition 12.17 *By a* **mixed strategy** *we mean an assignment of a probability (between* 0 *and* 1*, inclusive) to each possible strategy.*

Definition 12.18 *Suppose you and your opponent each choose a (mixed) strategy for a game. Then these strategies form a* **Nash equilibrum**[24] *if: your current strategy is optimal against her current strategy, and her current strategy is optimal against your current strategy.*

(p. 94) **Game Description** (The Big Wheel – Price Is Right): The **Big Wheel** consists of twenty numbers – 5 through 100 (i.e. five cents through a dollar), in increments of five. Three players compete, and the player who spins the closest to a dollar without going over advances to the Showcase Showdown.

The players spin in order. Each player spins once, and then either keeps the result or elects to spin a second time and add the two results. If the result is higher than \$1.00, the player is eliminated immediately. The winner is the player who spins the highest without going over. (If two or more players tie, they advance to a tiebreaker.)

In addition, players earn a bonus if they spin exactly a dollar – but we will ignore this.

(p. 101) **Game Description** (Hit Me (The Price Is Right)): The contestant plays a game of blackjack against the dealer, where the objective is to get a total of 21 without going over. The dealer plays as in ordinary blackjack: it deals two cards at random, and if it has a total of 16 or less it keeps drawing cards until it is over 16.

The contestant is shown six prizes along with six prices, each of which is $n\times$ the actual price of the item. Behind each prize is a card worth n. The contestant chooses prices one at a time and her hand is made up of these cards.

One of the prices will always be exactly right (so $n = 1$, and the card is ace, which in blackjack you may count as eleven), and one of them will be ten times the right price. If the contestant picks these two prizes first, she gets a blackjack (21) and wins no matter what the dealer has. Otherwise, she still has some opportunities to win.

(p. 102)

`https://www.youtube.com/watch?v=3EqBci6OQNg`

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[24] Named after John Forbes Nash, as depicted in the movie *A Beautiful Mind.*

(p. 102) **Game Description** (Bonkers (The Price Is Right)): The contestant is shown a prize whose price is four digits. She is then shown a board with a four digit price for the item. Each digit is wrong, and there are spaces to put paddles above and below each digit.

She must guess whether each digit is too high or too low, by placing paddles in the appropriate location and hitting a button (after which she gets the prize if her guess is right, and buzzed if it is wrong). She has thirty seconds and may guess as many times as she is physically able to.

(p. 103)

`https://www.youtube.com/watch?v=iZzBu5K_aBA`

(p. 106) **The Umbrella Problem.** One hundred guests attend a party. It is raining, and they all bring umbrellas to the party. All of their umbrellas are different from each other.

At the end of the party, the host hands umbrellas back to the guests at random. What is the probability that *nobody* gets their own umbrella back?

(p. 108) **Game Description** (Switcheroo – The Price Is Right): The contestant is shown five prizes – four very small prizes and a car. The contestant is also shown the price of each with the tens digits removed. The tens digits of the prizes are all different from each other, and the contestant is given five blocks with the tens digits written on them.

The contestant must match the blocks to the removed tens digits. She has thirty seconds to do this, after which she is shown how many prizes she has matched correctly (but not *which* prizes she has matched correctly). She may then switch around the blocks if she likes. She wins all prizes which she prices correctly.