Midterm Examination, Algebraic Number Theory (Math 788p), Frank Thorne (thorne@math.sc.edu)

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(1) Prove or disprove.

Let G be a finite group. Then there exists a field K, Galois over $\mathbb{Q}$, with $\mathrm{Gal}(K/\mathbb{Q}) \cong G$.

If true, further explain how to construct a K for any G.

(2) Are there infinitely many real quadratic fields which are UFDs? (Prove or disprove.)

(3) For any odd $\ell \geq 3$, and any $r \geq 1$ and $X \geq 1$, define $\delta(X, \ell, r)$ to be the proportion of imaginary quadratic fields K, with $|\mathrm{Disc}(K)| < X$, and $\mathrm{Cl}(K) \cong (\mathbb{Z}/\ell)^r \times H$ for some abelian group H with order coprime to ℓ.

Prove, for each fixed ℓ and r, that $\delta(X, \ell, r)$ converges to a positive limit $\delta(\ell, r)$, and determine this limit.

(4) Let $K/\mathbb{Q}$ be a finite Galois extension, and let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of $\mathrm{Gal}(K/\mathbb{Q})$.

For each prime p, define $\sigma_p \in \mathrm{Gal}(K/\mathbb{Q})$ as follows: First choose arbitrarily any prime ideal $\mathfrak{p}$ of K above p. Then, for all but finitely many primes p there exists a unique $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\mathfrak{p}) = \mathfrak{p}$, and $\sigma(x) - x^{\mathcal{N}(p)} \in \mathfrak{p}$ for all $x \in \mathcal{O}_K$.

For $\Re(s) > 1$, define a function $f(s)$ by

$$f(s) = \prod_p \det(1 - \rho(\sigma_p)p^{-s}),$$

where $1 - \rho(\sigma_p)p^{-s}$ is an endomorphism of $\mathbb{C}^n$, and the product is over all primes for which σ_p is uniquely defined, up to the choice of $\mathfrak{p}$.

Prove that the meromorphic continuation of $f(s)$ to all of $\mathbb{C}$ has at most finitely many poles.